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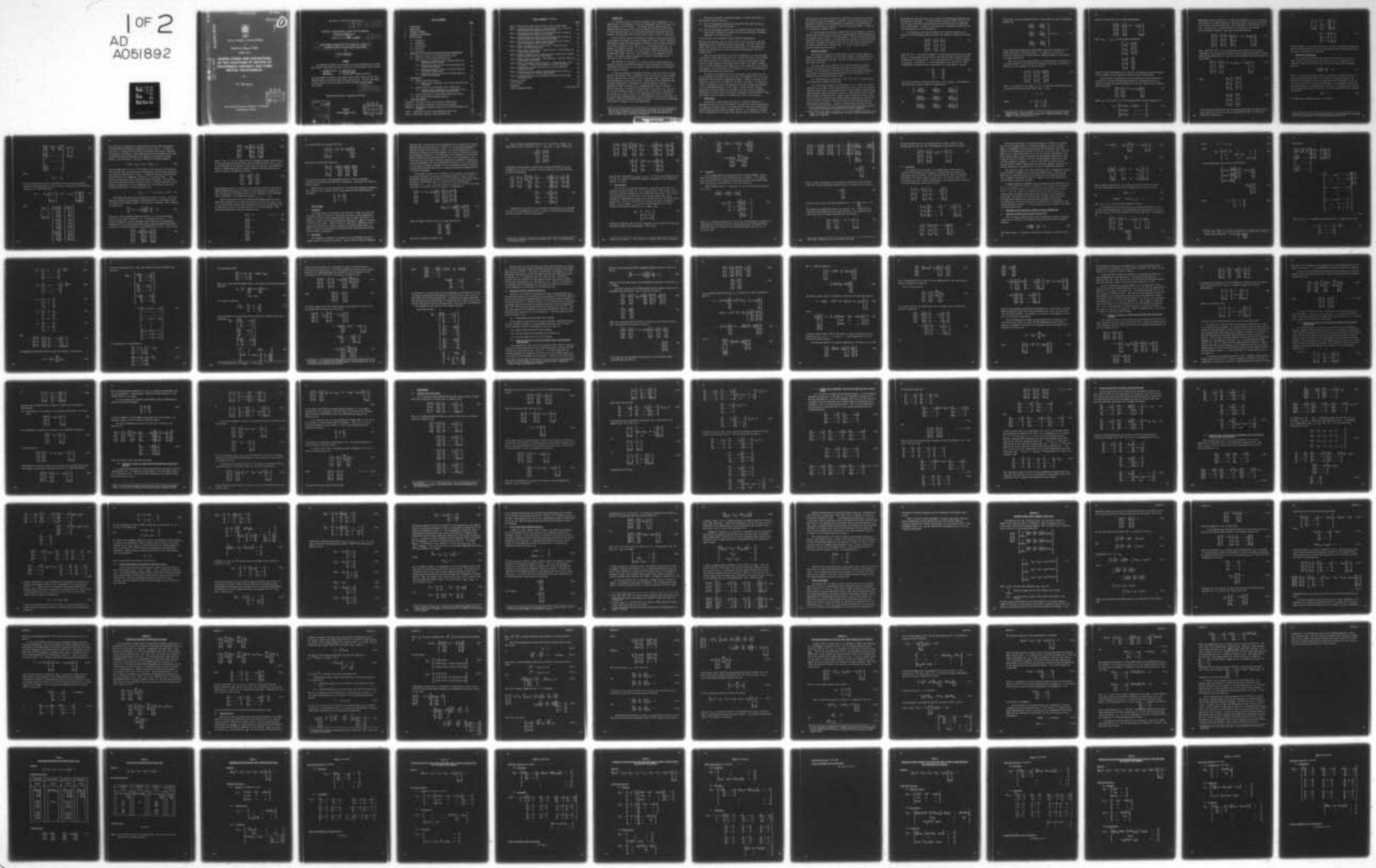

ROYAL AIRCRAFT ESTABLISHMENT

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DIVERS FORMS AND DERIVATIONS OF THE EQUATIONS OF MOTION OF DEFORMABLE AIRCRAFT AND THEIR MUTUAL RELATIONSHIP

by

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SUMMARY

The equations of motion of an aircraft, for small perturbations from flight with constant linear and zero angular velocities, are developed in detail using:-

constant-velocity	or	body-fixed axes,
encastré	or	free-free modes,
displacement	or	velocity body freedom coordinates.

The relationship is clearly stated between these various forms; and with other proposed forms, in particular those using mean-body axes. The whole development is kept consistent, as far as possible, with the Hopkin notation scheme (R & M 3562).

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1 INTRODUCTION

An essential feature of effective research is good communication. For communication to be successful it has to be unambiguous, and both attractive and understandable to the 'recipient'. One wonders how much good work has been largely wasted not because it has not been reported clearly but because those who could have made good use of it have been deterred by approaches which were strange to them and which they could not easily relate to their own experience.

Several different groups of aeronautical research scientists have been led to investigate the dynamics of aircraft taking some account of their flexibility. These groups include those whose primary concerns are active control, stability, ride control, structural loads, flutter etc. In this particular field - the dynamics of deformable aircraft - it has long been recognised that there are serious communication barriers. In a recent paper[1] Taylor and Woodcock each independently sought to provide a clear statement of the fundamentals of this subject. The present paper is intended as a sequel to that paper[1] and in particular to Part II of that paper[1].

The objective of this paper is to study in detail various forms of the equations of motion, for small perturbations from a datum motion, establish the relationship between them, and, if possible, demonstrate how one can transform from one form to another.

To study the dynamics of deformable aircraft we need the concepts of datum motion of the aircraft, and the undeformed state of the aircraft, where throughout the datum motion the aircraft is in the undeformed state. Departures from the datum motion are called perturbations. Departures from the undeformed state are called deformations. Consequently we need two frames of reference - a datum-motion frame of reference and an undeformed-state frame of reference - which are such that, if there are no perturbations the aircraft is at rest relative to the datum-motion frame of reference, and, if there are no deformations the aircraft is at rest relative to the undeformed-state frame of reference*. The perturbations comprise therefore (a) the translations and rotations which are necessary to make the datum-motion frame of reference coincide with the undeformed-state frame of reference, and (b) the deformations.

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* One could of course treat all perturbations in the same way as the deformations, and use only one frame of reference in their description, but from several points of view, and in particular consideration of large perturbations, the above procedure seems to be the most attractive.

We restrict ourselves to Lagrangian methods. Of these there seem to be three possible types of approach:

(i) The use of Lagrange's equation for an inertial frame which is taken as the datum-motion frame of reference.

(ii) The use of Lagrange's equation for a non-inertial frame (the undeformed-state frame of reference) in conjunction with the principles of momentum equations for the whole aircraft.

(iii) The same as (ii) followed by some coordinate transformation.

With the first approach there is usually little doubt as to which is the most appropriate choice of axes within the reference frame. The axes, within that frame, which coincide with a particularly significant set of body-fixed axes, such as the principal axes of inertia, during the datum motion, would be a suitable choice.

When the second approach is used there is something to be said for taking a non-inertial frame whose position etc is of interest in itself. We have therefore taken a certain set of body-fixed axes to define the non-inertial frame. To do so involves imposing certain minor restrictions on the modes of deformation. The third approach can be thought of as a way of removing, to all intents and purposes, these restrictions at the expense of a loss in the significance of the body-freedom generalised coordinates.

The notation and nomenclature used is consistent, apart from one or two noted exceptions, with that of Part II of Ref 1, and therefore also, to the same degree, with that of Hopkin's comprehensive scheme[2]. Attention is drawn to the Glossary of terms and List of symbols at the end of this paper. We confine ourselves to dimensional forms of the equations of motion and leave any normalisation or non-dimensionalisation to the reader. A further paper[16], which is essentially a particularisation of the present work for the case of symmetric perturbations in heave, pitch, fore and aft translation, and one deformation mode, does however also give suggested non-dimensional forms of the equations.

2 DATUM MOTION

We postulate a datum motion, with respect to which the dynamics of a deformable aircraft are to be investigated. One particular type of datum motion is considered: straight flight, not necessarily level, with the aircraft having constant linear velocity and zero angular velocity. It is assumed, as desired, that no deformation of the aircraft takes place during the datum motion. This

implies that all the forces acting on the aircraft are constant* and in particular, therefore, that the atmosphere is uniform. The only forces acting on the aircraft are assumed to be the aerodynamic and propulsive, gravitational and structural forces; with the addition, in the case of ground contact, of what we call upholding (or support) forces.

As a basic frame of reference a constant-velocity axes system is used with origin at the aircraft centre of gravity in the datum motion and axes parallel to its principal axes of inertia in the datum motion. They are therefore body-fixed axes during the datum motion; in fact the axes which Hopkin[2] calls datum-attitude earth axes. Throughout this paper we will use the term constant-velocity axes for these particular axes. The subscript f is used to denote values of quantities during the datum motion.

3 DEGREES OF FREEDOM

The aircraft is assumed to be semi-rigid, ie having a finite number of degrees of freedom in addition to its six body freedoms. Two representations of any possible perturbations are used. They are not exactly equivalent but to first order they agree.

The first, chosen for convenience in derivations using the constant-velocity axes (alias the datum-attitude earth axes), can be visualised as follows. The transformation of the aircraft from its datum motion position at any instant to its perturbed position and shape at the same instant can be achieved by the following successive steps:

(i) Translations, as a rigid body, $x_1^{(c)}$, $y_1^{(c)}$, $z_1^{(c)}$ in the directions of the respective constant-velocity axes.

(ii) Successive rotations, as a rigid body, ψ, θ, ϕ about the carried axes** Oz, Oy, Ox where Oxyz are body-fixed axes, with origin at the particle (reference point) which is coincident with the aircraft centre of gravity during the datum motion, whose orientation is fixed in a small portion of the aircraft which includes the reference point, which is either essentially rigid or otherwise such that the axes will always remain mutually perpendicular. The orientation of the axes is chosen so that they coincide with the constant-velocity axes during the datum motion. A frame of reference which coincides with the

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* We are assuming also that the aircraft's mass and mass distribution are constant and so neglecting the changes produced by fuel consumption.

** By carried axes is meant the position of the axes following the previous rotations. It is important to remember that the order of these rotations is not commutative in general.

body-fixed axes after this step will be called the no-deformation-body-fixed axes. The bracketed superscript n is used to denote reference to them. They are, as will be seen, in general only body-fixed as long as the aircraft is undeformed from its datum shape. It should be noted that their relevance is limited to this first representation of the perturbations.

(iii) Deformation such that the position of a particle relative to the origin of the no-deformation-body-fixed axes, and referred to those axes is given by

$$\begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix} = \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + R\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \tag{1}$$

where there are n deformational degrees of freedom represented here by the modal matrix R whose elements are functions of the particle being considered and are independent of the datum motion. For a given datum motion R will be a function of (x_f, y_f, z_f). The function R is also constrained to be such that for small perturbations the body-fixed axes remain mutually at right angles. Writing

$$R = \begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \\ c_1 & \cdots & c_n \end{bmatrix} \tag{2}$$

the direction cosines of the body-fixed axes, in the $O_n x_n y_n z_n$ (no-deformation-body-fixed axes) reference frame, are proportional to

$$\left(1 + \sum\left(\frac{\partial a_i}{\partial x_f}\right)_0 q_i\ ,\quad \sum\left(\frac{\partial b_i}{\partial x_f}\right)_0 q_i\ ,\quad \sum\left(\frac{\partial c_i}{\partial x_f}\right)_0 q_i\right)\ ,$$

$$\left(\sum\left(\frac{\partial a_i}{\partial y_f}\right)_0 q_i\ ,\quad 1 + \sum\left(\frac{\partial b_i}{\partial y_f}\right)_0 q_i\ ,\quad \sum\left(\frac{\partial c_i}{\partial y_f}\right)_0 q_i\right)\ ,$$

and

$$\left(\sum\left(\frac{\partial a_i}{\partial z_f}\right)_0 q_i\ ,\quad \sum\left(\frac{\partial b_i}{\partial z_f}\right)_0 q_i\ ,\quad 1 + \sum\left(\frac{\partial c_i}{\partial z_f}\right)_0 q_i\right)\ ,$$

respectively; and so they remain mutually at right angles for small perturbations provided that

$$\left.\begin{aligned}\left(\frac{\partial a_i}{\partial y_f}\right)_0 &= -\left(\frac{\partial b_i}{\partial x_f}\right)_0 \\ \left(\frac{\partial b_i}{\partial z_f}\right)_0 &= -\left(\frac{\partial c_i}{\partial y_f}\right)_0 \\ \left(\frac{\partial c_i}{\partial x_f}\right)_0 &= -\left(\frac{\partial a_i}{\partial z_f}\right)_0\end{aligned}\right\} \text{ for all } i \ . \tag{3}$$

A more complicated deformational representation than that of equation (1), or further constraints on the modal matrix R in addition to (3), would be necessary to ensure that the body-fixed axes remained mutually at right angles for larger perturbations.

Thus, with this first representation of the perturbation, the position of a particle relative to the origin of the constant-velocity axes and referred to those axes is (*cf.* Appendix A of Part II of Ref 1)

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} = \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} + S^T \begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix} \tag{4}$$

where S , a function of the angles ϕ, θ, ψ , is the axes transformation matrix (or attitude deviation matrix) of Refs 1 and 2. Since

$$S \approx I - A_\phi \tag{5}$$

where*

$$A_\phi = \begin{bmatrix} 0 & -\psi & \theta \\ \psi & 0 & -\phi \\ -\theta & \phi & 0 \end{bmatrix} \tag{6}$$

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* Following Refs 1 and 2 the symbol A is used to denote a particular skew-symmetric matrix formed from the elements of a column matrix whose leading element is used as a subscript to A .

equation (4) can be written, for small perturbations

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} \approx \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + [R \quad I \quad -A_{x_f}] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \tag{7}$$

where $q_{n+1}, \ldots, q_{n+6}$ are body freedom coordinates:-

$$\begin{bmatrix} q_{n+1} \\ q_{n+2} \\ q_{n+3} \end{bmatrix} = \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} q_{n+4} \\ q_{n+5} \\ q_{n+6} \end{bmatrix} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} . \tag{9}$$

However a better approximation, (54), than (7), is required for certain purposes in the derivation of the equations of motion for small perturbations.

The alternative representation of any perturbations from the datum motion condition is to define the deformation relative to the body-fixed axes (see Glossary of terms) such that the position of a particle relative to the reference point is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + (R - R_0 + A_{x_f} P_q) \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{bmatrix} \tag{10}$$

where R_0 is the value of R at the reference point, and (*cf.* equation (2))

$$P_q = \begin{bmatrix} (\partial c_1/\partial y_f)_0 & (\partial c_2/\partial y_f)_0 & \cdots\cdots \\ (\partial a_1/\partial z_f)_0 & \cdots\cdots\cdots\cdots & \\ (\partial b_1/\partial x_f)_0 & \cdots\cdots\cdots\cdots & \end{bmatrix} . \tag{11}$$

Remembering that the elements of R satisfy the condition (3), it is easily seen that equation (10) always represents a perturbation which is precisely a deformation relative to the body-fixed axes - the displacement of the body-fixed axes is zero. The position of a particle relative to the origin of the constant-velocity axes and referred to those axes is therefore given by

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} = \begin{bmatrix} \hat{x}_1^{(c)} \\ \hat{y}_1^{(c)} \\ \hat{z}_1^{(c)} \end{bmatrix} + \hat{S}^T \left\{ \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + (R - R_0 + A_{xf}P_q) \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{bmatrix} \right\} \tag{12}$$

where $(\hat{x}_1^{(c)}, \hat{y}_1^{(c)}, \hat{z}_1^{(c)})$ are the translations, and $(\hat{\phi}, \hat{\theta}, \hat{\psi})$ are the rotations (according to the standard Euler procedure) which transform the constant-velocity axes into the body-fixed axes. Thus for small perturbations this second representation gives

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} \approx \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + [(R - R_0 + A_{xf}P_q) \quad I \quad -A_{xf}] \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \tag{13}$$

where

$$\begin{bmatrix} \hat{q}_{n+1} \\ \hat{q}_{n+2} \\ \hat{q}_{n+3} \end{bmatrix} = \begin{bmatrix} \hat{x}_1^{(c)} \\ \hat{y}_1^{(c)} \\ \hat{z}_1^{(c)} \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} \hat{q}_{n+4} \\ \hat{q}_{n+5} \\ \hat{q}_{n+6} \end{bmatrix} = \begin{bmatrix} \hat{\phi} \\ \hat{\theta} \\ \hat{\psi} \end{bmatrix} . \tag{15}$$

It will be seen therefore that the two representations agree for small perturbations (equations (7) and (13)) when the relationship between the two sets of generalised coordinates is

$$\begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \approx \begin{bmatrix} I & 0 & 0 \\ -R_0 & I & 0 \\ -P_q & 0 & I \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 & 0 \\ R_0 & I & 0 \\ P_q & 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} . \quad (16)$$

There is however, no relationship between the two sets of generalised coordinates which will ensure agreement of the two representations when the perturbations are not small (ie agreement of (4) and (12)).

4 EQUATIONS OF EQUILIBRIUM

The constant-velocity axes (datum-attitude earth axes) are an inertial frame, since the datum motion has zero angular velocity. Consequently the equations of motion can be derived from the inertial frame form of Lagrange's *equation:-*

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{q}_i}\right) - \frac{\partial W}{\partial q_i} = \overline{Q_i} \quad (17)$$

where W is the kinetic energy relative to the frame of reference, and the $\overline{Q}_i$ are the total generalised* forces in the degrees of freedom obtained by the principle of virtual work. With the freedom specified by equations (4) and (7) it is easily seen that, for small perturbations, W is a quadratic form in the time derivatives $\dot{q}_i$ of the generalised coordinates. Consequently the existence of the datum motion requires that each of the total generalised forces* satisfies the condition

$$(\overline{Q_i})_f = 0 \quad (18)$$

as regards their unperturbed values. Now from (7)

* We have referred to the 'total generalised force' since subsequently we describe the various constituents as 'structural generalised force' etc.

$$\begin{bmatrix} \frac{\partial x_c^{(c)}}{\partial q_1} & \frac{\partial y_c^{(c)}}{\partial q_1} & \frac{\partial z_c^{(c)}}{\partial q_1} \\ \frac{\partial x_c^{(c)}}{\partial q_2} & \cdots\cdots\cdots & \\ \vdots & \cdots\cdots\cdots & \\ \frac{\partial x_c^{(c)}}{\partial q_{n+6}} & \cdots\cdots\cdots & \end{bmatrix} \approx \begin{bmatrix} R^T \\ I \\ A_{x_f} \end{bmatrix} \tag{19}$$

since

$$A_{x_f}^T = -A_{x_f} \tag{20}$$

and so the virtual work done by a distribution of force vectors $(\bar{e}_f,\bar{f}_f,\bar{g}_f)$ referred to the constant-velocity axes, acting on the particle whose location is (x_f,y_f,z_f), is, in the datum state,

$$[\delta q_1 \ \ldots \ \delta q_{n+6}] \begin{bmatrix} (\bar{Q}_1)_f \\ \vdots \\ (\bar{Q}_{n+6})_f \end{bmatrix} = [\delta q_1 \ldots \delta q_{n+6}] \sum \begin{bmatrix} R^T \\ I \\ A_{x_f} \end{bmatrix} \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} . \tag{21}$$

Thus

$$\begin{bmatrix} (\bar{Q}_1)_f \\ \vdots \\ (\bar{Q}_{n+6})_f \end{bmatrix} = \begin{bmatrix} \sum R^T \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} \\ \sum \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} \\ \sum A_{x_f} \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \sum R^T \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} \\ \begin{bmatrix} \bar{X}_f \\ \bar{Y}_f \\ \bar{Z}_f \end{bmatrix} \\ \begin{bmatrix} \bar{L}_f \\ \bar{M}_f \\ \bar{N}_f \end{bmatrix} \end{bmatrix} \tag{22}$$

where $(\bar{X}_f,\bar{Y}_f,\bar{Z}_f)$, $(\bar{L}_f,\bar{M}_f,\bar{N}_f)$ are respectively the overall forces and moments produced by the above mentioned distribution of force vectors. In flight the force vector on any particle may include aerodynamic, gravitational, propulsive and structural constituents. Other constituents such as the magnetic forces may be present but they are considered to be negligible as regards our present purpose. Thus the equation (18) can be rewritten as the matrix equation

$$[-(Q_i)_f + (G_i)_f + (P_i)_f + (E_i)_f] = 0 \tag{23}$$

where the expression in the [ ] is the element in the ith row of the matrix and the individual terms are respectively the aerodynamic, gravitational, propulsive, and structural contributions to $-(\bar{Q}_i)_f$. Expression for these, derived following (22) and with the assumptions of section 3.1 are given in Table 1. The structural generalised forces in particular merit some comment. The structure cannot exert any overall force or moment on itself - this is intuitively true and moreover is demonstrated to be so in Appendix A for the particular case of an isotropic elastic body. It follows therefore that

$$(E_i)_f = 0 \quad \text{for } i = (n+1),\ldots,(n+6) . \tag{24}$$

With body-fixed axes we can as demonstrated in Part II of Ref 1 derive the equations of motion using Lagrange's equations referred to a non-inertial frame in conjunction with equations based on the principle of momentum. These equations are respectively

$$\frac{\partial V_0}{\partial \hat{q}_i} + J_i + G_i + \frac{d}{dt}\left(\frac{\partial \hat{W}}{\partial \dot{\hat{q}}_i}\right) - \frac{\partial \hat{W}}{\partial \hat{q}_i} = \bar{\hat{Q}}_i \tag{25}$$

where V_0 is the centrifugal potential function, $\hat{W}$ is the kinetic energy relative to the frame of reference, G_i is the gyrostatic force, J_i is a certain coupling force between rotational body freedoms and the deformational freedoms, and the $\bar{\hat{Q}}_i$ are the generalised forces obtained regarding the frame of reference as stationary during any virtual displacement (*cf.* Ref 1, Part II); and

$$\begin{bmatrix} \bar{\hat{X}} \\ \bar{\hat{Y}} \\ \bar{\hat{Z}} \end{bmatrix} = \sum \delta m \left(\begin{bmatrix} \dot{u}_m \\ \dot{v}_m \\ \dot{w}_m \end{bmatrix} + A_p \begin{bmatrix} u_m \\ v_m \\ w_m \end{bmatrix} \right) \tag{26}$$

$$\begin{bmatrix} \bar{\hat{L}} \\ \bar{\hat{M}} \\ \bar{\hat{N}} \end{bmatrix} = \sum \delta m A_x \left(\begin{bmatrix} \dot{u}_m \\ \dot{v}_m \\ \dot{w}_m \end{bmatrix} + A_p \begin{bmatrix} u_m \\ v_m \\ w_m \end{bmatrix} \right) \tag{27}$$

where $\bar{\hat{X}}$ etc, $\bar{\hat{L}}$ etc, are respectively the resolutes along the body-fixed axes of the total applied force and torque about the reference point, and u_m etc, are the resolutes of the velocity of the particle at (x,y,z). For perturbations from our assumed datum motion the angular velocity of the body-fixed axes is given by (see Part II, Appendix A of Ref 1)

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = Q_{\hat{\phi}} \begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\psi}} \end{bmatrix} \approx \begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\psi}} \end{bmatrix} \tag{28}$$

The perturbed position of a particle will be taken to be given by equation (10) which satisfies the condition, necessary for the application of equation (25), that the position of the reference point and the orientation of the body-fixed axes is independent of the degrees of freedom q_i (i = 1, ..., n).

Since the linear velocity is constant and the angular velocity is zero during the datum motion it is found (see section 6.3 of Part II of Ref 1) that all the terms on the left-hand side of equation (25) and on the right-hand sides of equations (26) and (27) are zero when there are no perturbations. Consequently the datum motion equilibrium conditions are

$$(\bar{\hat{Q}}_i)_f = 0 \qquad i = 1,\ldots,n \tag{29}$$

$$\begin{bmatrix} \bar{X}_f \\ \bar{Y}_f \\ \bar{Z}_f \end{bmatrix} = 0 \tag{30}$$

$$\begin{bmatrix} \bar{L}_f \\ \bar{M}_f \\ \bar{N}_f \end{bmatrix} = 0 \quad . \tag{31}$$

It is easily shown (*cf.* equation (10)) that

$$\begin{bmatrix} (\bar{\hat{Q}}_1)_f \\ \vdots \\ (\bar{\hat{Q}}_n)_f \end{bmatrix} = \sum (R^T - R_0^T - P_q^T A_{xf}) \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} \tag{32}$$

which can be alternatively written as

$$\begin{bmatrix} (\bar{\hat{Q}}_1)_f \\ \vdots \\ (\bar{\hat{Q}}_n)_f \end{bmatrix} = \sum R^T \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} - R_0^T \begin{bmatrix} \bar{X}_f \\ \bar{Y}_f \\ \bar{Z}_f \end{bmatrix} - P_q^T \begin{bmatrix} \bar{L}_f \\ \bar{M}_f \\ \bar{N}_f \end{bmatrix} . \tag{33}$$

We can as before express these generalised forces as a sum of aerodynamic, propulsive, gravitational and structural contributions. Expressions for these are given in Table 2.

Finally we note that the second form of the datum motion equations (equations (29), (30) *and* (31)) is merely the first form (equation (18)) premultiplied by the matrix

$$\begin{bmatrix} I & -R_0^T & -P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} . \tag{34}$$

5 APPLIED FORCES

5.1 Assumptions

The primary object of this study is the aircraft in flight well away from the ground. When the aircraft is flying near the ground certain assumptions in respect of the form of the aerodynamic forces cannot be made, while when it is actually in contact with the ground additional upholding (or support) forces have to be taken into account. These two cases are considered in Appendices B and C respectively while here we restrict ourselves to situations where there is no effect at all of the near presence of the earth other than gravity and the fact that the flight is atmospheric.

5.2 Aerodynamic

The atmosphere is assumed to be uniform, and the aerodynamic forces are assumed to have no hereditary constituent. Both of these are obvious departures

from the truth. The first will rarely produce much error provided the 'uniform' atmosphere has the same properties - density etc - as the actual atmosphere at some mean altitude of the aircraft's motion. For a perturbation which is a maintained oscillation the second assumption is in effect eliminated by allowing the aerodynamic coefficients to be functions of the frequency of the oscillation. Moreover Woodcock and Lawrence have shown[4,5] that a good approximation to any calculated oscillatory motion (decaying, maintained or growing) is obtained provided the aerodynamics used is for maintained oscillations of the same frequency. In such cases one assumes a frequency for the aerodynamics, calculates the motion and then repeats the procedure until the assumed and calculated frequencies are in sufficient agreement.

The boundary condition to be satisfied by the air motion at the surface of the aircraft involves the velocity of the surface normal to itself. With the perturbation representation of equation (10) the slope vector of the body surface, referred to the body-fixed axes, is a function only of the generalised coordinates $\hat{q}_1, \ldots, \hat{q}_n$. Also the velocity of a particle, referred to the body-fixed axes, is (see Ref 1, Part II, section 6.2 and Appendix A)

$$\begin{bmatrix} u_m \\ v_m \\ w_m \end{bmatrix} = (R - R_0 + A_{x_f} P_q) \begin{bmatrix} \dot{\hat{q}}_1 \\ \vdots \\ \dot{\hat{q}}_n \end{bmatrix} + \hat{S} \left\{ \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + \begin{bmatrix} \dot{\hat{x}}_1^{(c)} \\ \dot{\hat{y}}_1^{(c)} \\ \dot{\hat{z}}_1^{(c)} \end{bmatrix} \right\} - \left(\hat{S} A_{\hat{x}_1^{(c)}} \hat{S}^T + A_x \right) Q_{\hat{\phi}} \begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\psi}} \end{bmatrix} - A_{\hat{p}} \hat{S} \begin{bmatrix} \hat{x}_1^{(c)} \\ \hat{y}_1^{(c)} \\ \hat{z}_1^{(c)} \end{bmatrix} \tag{35}$$

where the angular velocity vector of the body-fixed axes is

$$\begin{bmatrix} \hat{p} \\ \hat{q} \\ \hat{r} \end{bmatrix} = Q_{\hat{\phi}} \begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\psi}} \end{bmatrix} \tag{36}$$

and {x,y,z} is given by equation (10).

Thus, for small perturbations (where $\hat{S} \approx I - A_{\hat{\phi}}$, $Q_{\dot{\hat{\phi}}} \approx I + O(\hat{\phi})$) , the normal wash* will be a function of $(\hat{q}_1, \ldots, \hat{q}_n)$, $(\dot{\hat{q}}_1, \ldots, \dot{\hat{q}}_n)$, $(\dot{\hat{\phi}}, \dot{\hat{\theta}}, \dot{\hat{\psi}})$, and

$$A_{u_f} \begin{bmatrix} \hat{\phi} \\ \hat{\theta} \\ \hat{\psi} \end{bmatrix} + \begin{bmatrix} \dot{\hat{x}}_1^{(c)} \\ \dot{\hat{y}}_1^{(c)} \\ \dot{\hat{z}}_1^{(c)} \end{bmatrix} .$$

Consequently, having assumed no ground and no hereditary effects, the local aerodynamic force vector, referred to the body-fixed axes, will have the form, for small perturbations:

$$\begin{bmatrix} e \\ f \\ g \end{bmatrix} \approx \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} + \sum_{j=1}^{n} \begin{bmatrix} \hat{e}_j \\ \hat{f}_j \\ \hat{g}_j \end{bmatrix} \hat{q}_j + \begin{bmatrix} \hat{e}_{n\dot{+}1} & \ldots & \hat{e}_{n\dot{+}3} \\ \hat{f}_{n\dot{+}1} & \ldots\ldots\ldots & \\ \hat{g}_{n\dot{+}1} & \ldots\ldots\ldots & \end{bmatrix} \left\{ \begin{bmatrix} \dot{\hat{q}}_{n+1} \\ \dot{\hat{q}}_{n+2} \\ \dot{\hat{q}}_{n+3} \end{bmatrix} + A_{u_f} \begin{bmatrix} \hat{q}_{n+4} \\ \hat{q}_{n+5} \\ \hat{q}_{n+6} \end{bmatrix} \right\}$$

$$+ \begin{bmatrix} \hat{e}_{n\dot{+}4} & \ldots & \hat{e}_{n\dot{+}6} \\ \hat{f}_{n\dot{+}4} & \ldots\ldots\ldots & \\ \hat{g}_{n\dot{+}4} & \ldots\ldots\ldots & \end{bmatrix} \begin{bmatrix} \dot{\hat{q}}_{n+4} \\ \dot{\hat{q}}_{n+5} \\ \dot{\hat{q}}_{n+6} \end{bmatrix} . \qquad (37)$$

Similarly we find that with the alternative representation of the deformation the local aerodynamic force vector, referred to the constant-velocity axes, will have the form, for small perturbations:

---

* Terms such as downwash, sidewash are commonly used. This is a generalisation to suit any surface.

$$
\begin{bmatrix} e^{(c)} \\ f^{(c)} \\ g^{(c)} \end{bmatrix} \approx \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} + \sum_{j=1}^{n} \begin{bmatrix} e_j \\ f_j \\ g_j \end{bmatrix} q_j + \begin{bmatrix} e_{\dot{n+1}} & \cdots & e_{\dot{n+3}} \\ f_{\dot{n+1}} & \cdots\cdots\cdots \\ g_{\dot{n+1}} & \cdots\cdots\cdots \end{bmatrix} \left\{ \begin{bmatrix} \dot{q}_{n+1} \\ \dot{q}_{n+2} \\ \dot{q}_{n+3} \end{bmatrix} + A_{u_f} \begin{bmatrix} q_{n+4} \\ q_{n+5} \\ q_{n+6} \end{bmatrix} \right\}
$$

$$
- A_{e_f} \begin{bmatrix} q_{n+4} \\ q_{n+5} \\ q_{n+6} \end{bmatrix} + \begin{bmatrix} e_{\dot{n+4}} & e_{\dot{n+5}} & e_{\dot{n+6}} \\ f_{\dot{n+4}} & \cdots\cdots\cdots \\ g_{\dot{n+4}} & \cdots\cdots\cdots \end{bmatrix} \begin{bmatrix} \dot{q}_{n+4} \\ \dot{q}_{n+5} \\ \dot{q}_{n+6} \end{bmatrix} . \tag{38}
$$

Note that the coefficients $\hat{e}_j$, $\hat{e}_{\dot{n+1}}$, e_j, $e_{\dot{n+1}}$ etc in the above expressions (37) and (38) are in general differential operators (polynomials in the differential operator $D = d/dt$).

5.3 Gravitational

One can, for all intents and purposes*, assume the gravitational force acting on a particle of mass δm to be a constant force δmg acting in the z_0 direction (vertically downwards) of the normal earth-fixed axes[2]. The orientation transformation from these axes to the constant-velocity axes can be achieved by a standard Euler sequence of successive rotations Ψ_f, Θ_f, Φ_f about carried axes. Thus if we denote the last column of the axes transformation matrix S by the symbol ℓ_ϕ, then the local gravitational force vector referred to the constant-velocity axes is $\delta mg\ell_{\Phi_f}$ where

$$
\ell_{\Phi_f} = \begin{bmatrix} -\sin\Theta_f \\ \sin\Phi_f \cos\Theta_f \\ \cos\Phi_f \cos\Theta_f \end{bmatrix} . \tag{39}
$$

Referred to body-fixed axes the vector components are not constant but have some terms which depend on the perturbations. Thus we have

---

* The rate of change of g with location is extremely small almost everywhere.

$$\begin{bmatrix} e_g \\ f_g \\ g_g \end{bmatrix} = \delta mg\hat{S}\ell_{\Phi_f} \approx \delta mg \left\{ \ell_{\Phi_f} + A_{\ell_{\Phi_f}} \begin{bmatrix} \hat{\phi} \\ \hat{\theta} \\ \hat{\psi} \end{bmatrix} \right\}$$

$$= \text{(say)} \begin{bmatrix} e_{gf} \\ f_{gf} \\ g_{gf} \end{bmatrix} + \sum_{j=n+4}^{n+6} \begin{bmatrix} \hat{e}_{gj} \\ \hat{f}_{gj} \\ \hat{g}_{gj} \end{bmatrix} \hat{q}_j \quad . \tag{40}$$

5.4 Propulsive

A very simple model of the propulsive forces has been assumed. Time no doubt will tell whether a more sophisticated one is required. It is assumed that the propulsive force acting on any particular particle has constant components (e_{pf}, f_{pf}, g_{pf}) in the direction of the body-fixed axes.

Since the modal matrix R (equations (1) and (2)) satisfies the condition (3), we have

$$x_f\left(\frac{\partial R}{\partial x_f}\right)_0 + y_f\left(\frac{\partial R}{\partial y_f}\right)_0 + z_f\left(\frac{\partial R}{\partial z_f}\right)_0$$

$$= -A_{x_f}P_q + \begin{bmatrix} x_f & 0 & 0 \\ 0 & y_f & 0 \\ 0 & 0 & z_f \end{bmatrix} \begin{bmatrix} \left(\frac{\partial a_1}{\partial x_f}\right)_0 & \cdots \\ \left(\frac{\partial b_1}{\partial y_f}\right)_0 & \cdots \\ \left(\frac{\partial c_1}{\partial z_f}\right)_0 & \cdots \end{bmatrix} \tag{41}$$

where P_q is given by equation (11); and so, with the deformation of equation (1) the position of a particle relative to the reference point (the origin of the body-fixed axes) and referred to the no-deformation-body-fixed axes is, for (x_f, y_f, z_f) small,

$$\begin{bmatrix} x^{(n)} \\ y^{(n)} \\ z^{(n)} \end{bmatrix} \equiv \begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix} - \begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix}_0 \approx \begin{bmatrix} x_f & 0 & 0 \\ 0 & y_f & 0 \\ 0 & 0 & z_f \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \left(\frac{\partial a_1}{\partial x_y}\right)_0 & \cdots & \left(\frac{\partial a_n}{\partial x_f}\right)_0 \\ \left(\frac{\partial b_1}{\partial y_f}\right)_0 & \cdots\cdots\cdots & \\ \left(\frac{\partial c_1}{\partial z_f}\right)_0 & \cdots\cdots\cdots & \end{bmatrix} \right)$$

$$- A_{x_f} P_q \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad . \tag{42}$$

Thus, for small perturbations, the orientation transformation from the body-fixed axes to the no-deformation-body-fixed axes is achieved by the rotations*

$$- P_q \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad . \tag{43}$$

The last term in (42) is the first approximation to $(S - I)\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where S is the appropriate transformation matrix (*cf.* equation (5)). Consequently the orientation transformation from the body-fixed axes to the constant-velocity axes is the result, for small perturbations of the rotation

$$- \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} - P_q \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = - [P_q \quad 0 \quad I] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \quad . \tag{44}$$

---

* The order is immaterial since the rotations are small.

We find therefore that the local propulsive force vector, referred to the constant-velocity axes, has, for the deformation of equation (4), the value

$$\begin{bmatrix} e_p^{(c)} \\ f_p^{(c)} \\ g_p^{(c)} \end{bmatrix} \approx \begin{bmatrix} e_{pf} \\ f_{pf} \\ g_{pf} \end{bmatrix} - A_{e_{pf}} [P_q \quad 0 \quad I] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} . \tag{45}$$

5.5 Structural

On a particle of the aircraft it is assumed there is what we call a structural force produced by the stresses in the material and possibly also other causes such as friction. This structural force is taken to be entirely determined by the current shape of the aircraft. Thus the two forms of the structural force vector, referred respectively to the constant-velocity axes and to the body-fixed axes, that we will use are:-

$$\begin{bmatrix} e_s^{(c)} \\ f_s^{(c)} \\ g_s^{(c)} \end{bmatrix} \approx S^T \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} + \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$\approx \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} & 0 & -A_{e_{sf}} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \tag{46}$$

$$\begin{bmatrix} e_s \\ f_s \\ g_s \end{bmatrix} \approx \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} + \begin{bmatrix} \hat{e}_{s1} & \cdots & \hat{e}_{sn} \\ \hat{f}_{s1} & \cdots\cdots & \\ \hat{g}_{s1} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{bmatrix} . \tag{47}$$

The various coefficients in the above expression are taken to be constant. That is we are assuming there is no structural damping. No really adequate theory of the damping of aircraft structures has been developed. One either neglects it and assumes that by doing so one will be 'on the safe side' - and this seems to be at least nearly always so; or else one includes coefficients in the final equations based on ground resonance test measurements and primitive theory. It may well be that for most, if not all, aircraft dynamic studies one or other of these procedures is adequate. It will be noted that we could here fairly easily introduce a simple representation of structural damping by replacing the constant coefficients of equations (46) and (47) by differential operators.

What exactly is meant by the term 'structural force' is perhaps made clearer by the analysis of Appendix A for an isotropic elastic body. Intuitively one knows that the structural forces on the aircraft will not produce any overall force or moment on the whole aircraft (*cf.* Appendix A), but there may well be forces tending to deform the aircraft.

It should perhaps be emphasised that the structural force coefficients e_{si}, $\hat{e}_{si}$ etc of equations (46) and (47), in addition to the datum motion structural force, will in general depend on the chosen unperturbed condition (*cf* equation (A-13)). Thus, for example, the coefficients for the ground resonance test situation will be different from those for the aircraft in level flight at a given speed. Nevertheless one will often have to assume, particularly when structural force information is obtained from experiments on the ground (*cf.* Appendix C), that the structural force coefficients are independent of the chosen unperturbed condition. In such circumstances there is usually little hope of doing anything better.

6 DERIVATION OF THE EQUATIONS OF MOTION FOR SMALL PERTURBATIONS

6.1 Equations of motion using constant-velocity axes

With our chosen constant-velocity axes the equations of motion, as already pointed out in section 4, can be derived using the inertial form of Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{q}_i}\right) - \frac{\partial W}{\partial q_i} = \bar{Q}_i \quad . \tag{48}$$

The kinetic energy W, relative to the frame of reference, is, using (7) given by

$$W \approx \tfrac{1}{2}\sum \delta m[\dot{q}_1 \;\; \ldots \;\; \dot{q}_{n+6}] \begin{bmatrix} R^T \\ I \\ A_{x_f} \end{bmatrix} [R \quad I \quad -A_{x_f}] \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_{n+6} \end{bmatrix} \tag{49}$$

and so

$$\frac{\partial W}{\partial q_i} = 0 \tag{50}$$

$$\frac{d}{dt}\begin{bmatrix} \frac{\partial W}{\partial \dot{q}_1} \\ \vdots \\ \frac{\partial W}{\partial \dot{q}_n} \end{bmatrix} = \sum \delta m \begin{bmatrix} R^T R & R^T & -R^T A_{x_f} \\ R & I & -A_{x_f} \\ A_{x_f} R & A_{x_f} & -(A_{x_f})^2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_{n+6} \end{bmatrix} . \tag{51}$$

Since, during the datum motion, the reference point is the centre of gravity and the constant-velocity axes are the principal axes of inertia, we have

$$\sum \delta m A_{x_f} = 0 \tag{52}$$

and

$$-\sum \delta m (A_{x_f})^2 = \mathrm{diag}\{I_x \quad I_y \quad I_z\} = I_n \tag{53}$$

where I_x, I_y, I_z are the principal moments of inertia.

To determine the virtual work, and hence the generalised forces, we need to take a better approximation, to the expression for the position of a particle, than that (equation (7)) which sufficed for the unperturbed state. From equation (4)

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} \approx \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + [R \quad I \quad -A_{x_f}] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} + A_\phi R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ B_{\phi\theta}^T \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} \tag{54}$$

where $$S \approx I - A_\phi + B_{\phi\theta} \tag{55}$$

with*

$$B_{\phi\theta} = \begin{bmatrix} -\frac{1}{2}(\psi^2+\theta^2) & 0 & 0 \\ \phi\theta & -\frac{1}{2}(\phi^2+\psi^2) & 0 \\ \phi\psi & \theta\psi & -\frac{1}{2}(\phi^2+\theta^2) \end{bmatrix} . \tag{56}$$

It can be shown that, for any vector $\{\bar{e}^{(c)}\bar{f}^{(c)}\bar{g}^{(c)}\}$,

$$\begin{bmatrix} [x_f y_f z_f] \dfrac{\partial B_{\phi\theta}}{\partial\phi} \\ [x_f y_f z_f] \dfrac{\partial B_{\phi\theta}}{\partial\theta} \\ [x_f y_f z_f] \dfrac{\partial B_{\phi\theta}}{\partial\psi} \end{bmatrix} \begin{bmatrix} \bar{e}^{(c)} \\ \bar{f}^{(c)} \\ \bar{g}^{(c)} \end{bmatrix} = A_{\bar{e}^{(c)}} A_{x_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + K_\phi^T A_{x_f} \begin{bmatrix} \bar{e}^{(c)} \\ \bar{f}^{(c)} \\ \bar{g}^{(c)} \end{bmatrix} \tag{57}$$

where $$K_\phi = \begin{bmatrix} 0 & -\psi & 0 \\ \psi & 0 & 0 \\ -\theta & 0 & 0 \end{bmatrix} . \tag{57a}$$

---

* We have $B_{\phi\theta} = \frac{1}{2}(A_\phi^2 + A_{\theta\phi})$ where the subscript $\theta\psi$ denotes the A matrix formed from the elements of $\{\theta\psi \; -\psi\phi \; \phi\theta\} \left(= -K_\phi \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \right)$.

Consequently,

$$\begin{bmatrix} \frac{\partial x_c^{(c)}}{\partial q_1} & \frac{\partial y_c^{(c)}}{\partial q_1} & \frac{\partial z_c^{(c)}}{\partial q_1} \\ \frac{\partial x_c^{(c)}}{\partial q_2} & \cdots\cdots\cdots & \\ \vdots & \cdots\cdots\cdots & \\ \frac{\partial x_c^{(c)}}{\partial q_{n+6}} & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} \bar{e}^{(c)} \\ \bar{f}^{(c)} \\ \bar{g}^{(c)} \end{bmatrix} \approx \begin{bmatrix} R^T \\ I \\ A_{x_f} \end{bmatrix} \begin{bmatrix} \bar{e}^{(c)} \\ \bar{f}^{(c)} \\ \bar{g}^{(c)} \end{bmatrix}$$

$$+ \begin{bmatrix} [q_1 \;\; \ldots \;\; q_{n+6}] \; [0 \quad 0 \quad -A_{a_1}]^T \\ \ldots\ldots \qquad [0 \quad 0 \quad -A_{a_2}]^T \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [q_1 \;\; \ldots \;\; q_{n+6}] \; [0 \quad 0 \quad -A_{a_n}]^T \\ 0 \\ 0 \\ 0 \\ [q_1 \;\; \ldots \;\; q_{n+6}] \; [R_1 \quad 0 \quad T_1]^T \\ \ldots\ldots \qquad [R_2 \quad 0 \quad T_2]^T \\ \ldots\ldots \qquad [R_3 \quad 0 \quad T_3]^T \end{bmatrix} \begin{bmatrix} \bar{e}^{(c)} \\ \bar{f}^{(c)} \\ \bar{g}^{(c)} \end{bmatrix}$$

...... (58)

where a_i, b_i, c_i are elements of the modal matrix R (equation (2)), and

$$R_1 = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ -c_1 & & \ldots\ldots & -c_n \\ b_1 & & \ldots\ldots & b_n \end{bmatrix} = \frac{\partial A_\phi}{\partial \phi} R \qquad (59)$$

$$R_2 = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ 0 & \cdots\cdots & & 0 \\ -a_1 & \cdots\cdots & & -a_n \end{bmatrix} = \frac{\partial A_\phi}{\partial \theta} R \tag{60}$$

$$R_3 = \begin{bmatrix} -b_1 & -b_2 & \cdots & -b_n \\ a_1 & \cdots\cdots & & a_n \\ 0 & \cdots\cdots & & 0 \end{bmatrix} = \frac{\partial A_\phi}{\partial \psi} R \tag{61}$$

$$T_1 = \begin{bmatrix} 0 & y_f & z_f \\ -y_f & 0 & 0 \\ -z_f & 0 & 0 \end{bmatrix} \tag{62}$$

$$T_2 = \begin{bmatrix} y_f & -x_f & 0 \\ 0 & 0 & z_f \\ 0 & -z_f & 0 \end{bmatrix} \tag{63}$$

$$T_3 = \begin{bmatrix} z_f & 0 & -x_f \\ 0 & z_f & -y_f \\ 0 & 0 & 0 \end{bmatrix} . \tag{64}$$

With

$$\begin{bmatrix} \bar{e}^{(c)} \\ \bar{f}^{(c)} \\ \bar{g}^{(c)} \end{bmatrix} = \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots\cdots & \\ \bar{g}_1 & \cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \tag{65}$$

the generalised forces are therefore, using the principal of virtual work,

$$\bar{Q}_i = (\bar{Q}_i)_f + \sum_{j=1}^{n+6} \bar{Q}_{ij} q_j \tag{66}$$

where the expressions for the $(\bar{Q}_i)_f$ have already been given (equation (22)) and where

$$
[\bar{Q}_{ij}] = \begin{bmatrix}
\sum R^T \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots\cdots & \\ \bar{g}_1 & \cdots\cdots & \end{bmatrix} \\[2ex]
\sum \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots\cdots & \\ \bar{g}_1 & \cdots\cdots & \end{bmatrix} \\[2ex]
\sum A_{x_f} \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots\cdots & \\ \bar{g}_1 & \cdots\cdots & \end{bmatrix}
\end{bmatrix}
$$

$$
+ \sum \begin{bmatrix}
[\bar{e}_f \;\; \bar{f}_f \;\; \bar{g}_f]\,[0 \;\; 0 \;\; -A_{a_1}] \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
[\cdots\cdots]\,[0 \;\; 0 \;\; -A_{a_n}] \\
0 \\
0 \\
0 \\
[\bar{e}_f \;\; \bar{f}_f \;\; \bar{g}_f]\,[R_1 \;\; 0 \;\; T_1] \\
[\cdots\cdots]\,[R_2 \;\; 0 \;\; T_2] \\
[\cdots\cdots]\,[R_3 \;\; 0 \;\; T_3]
\end{bmatrix} . \tag{67}
$$

This expression can be simplified for

$$
\begin{bmatrix}
[\bar{e}_f \;\; \bar{f}_f \;\; \bar{g}_f] & R_1 \\
[\cdots\cdots\cdots] & R_2 \\
[\cdots\cdots\cdots] & R_3
\end{bmatrix} = - A_{\bar{e}_f} R \tag{68}
$$

$$
- \begin{bmatrix}
[\bar{e}_f \;\; \bar{f}_f \;\; \bar{g}_f] & A_{a_1} \\
[\cdots\cdots\cdots] & A_{a_2} \\
\cdots\cdots\cdots & \cdots \\
[\cdots\cdots\cdots] & A_{a_n}
\end{bmatrix} = R^T A_{\bar{e}_f} \tag{69}
$$

and (*cf* equation (57))*

$$\begin{bmatrix} [\bar{e}_f & \bar{f}_f & \bar{g}_f] & T_1 \\ [\ldots\ldots\ldots\ldots & &] & T_2 \\ [\ldots\ldots\ldots\ldots & &] & T_3 \end{bmatrix} = A_{x_f} A_{\bar{e}_f} + C_{\bar{e}_f x_f} \tag{70}$$

where C_{ex} is the lower triangular matrix, with zeroes on the principal diagonal, such that

$$C_{ex} - C_{ex}^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix} [e\ f\ g] - \begin{bmatrix} e \\ f \\ g \end{bmatrix} [x\ y\ z]$$

$$= A_e A_x - A_x A_e \quad . \tag{71}$$

It is easily be seen that

$$\sum C^T_{\bar{e}_f x_f} = \begin{bmatrix} 0 & -\bar{N}_f & \bar{M}_f \\ 0 & 0 & -\bar{L}_f \\ 0 & 0 & 0 \end{bmatrix} \tag{72}$$

and so the matrix of the generalised force coefficients (equation (67)) can be rewritten as

$$[\bar{Q}_{ij}] = \begin{bmatrix} \sum R^T \begin{bmatrix} \bar{e}_1 & \ldots & \bar{e}_{n+6} \\ \bar{f}_1 & \ldots\ldots\ldots \\ \bar{g}_1 & \ldots\ldots\ldots \end{bmatrix} \\ \sum \begin{bmatrix} \bar{e}_1 & \ldots & \bar{e}_{n+6} \\ \bar{f}_1 & \ldots\ldots\ldots \\ \bar{g}_1 & \ldots\ldots\ldots \end{bmatrix} \\ \sum A_{x_f} \begin{bmatrix} \bar{e}_1 & \ldots & \bar{e}_{n+6} \\ \bar{f}_1 & \ldots\ldots\ldots \\ \bar{g}_1 & \ldots\ldots\ldots \end{bmatrix} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \sum R^T A_{\bar{e}_f} \\ 0 & 0 & 0 \\ -\sum A_{\bar{e}_f} R & 0 & \sum A_{\bar{e}_f} A_{x_f} + \begin{bmatrix} 0 & -\bar{N}_f & \bar{M}_f \\ 0 & 0 & -\bar{L}_f \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \quad . \tag{73}$$

---

* It can be shown that $[e\ f\ g] A_x K_\phi = -[\phi\ \theta\ \psi] C_{ex}$.

Certain of the above terms can be expressed in terms of translational and rotational force coefficients. The overall translational forces on the aircraft, referred to the constant-velocity axes, produced by the distribution of local force vectors $\{\bar{e}^{(c)}, \bar{f}^{(c)}, \bar{g}^{(c)}\}$, is given by (*cf* equation (65))

$$\begin{bmatrix} \bar{X}^{(c)} \\ \bar{Y}^{(c)} \\ \bar{Z}^{(c)} \end{bmatrix} = \sum \begin{bmatrix} \bar{e}^{(c)} \\ \bar{f}^{(c)} \\ \bar{g}^{(c)} \end{bmatrix} = \text{(say)} \begin{bmatrix} \bar{X}_f \\ \bar{Y}_f \\ \bar{Z}_f \end{bmatrix} + \sum_{j=1}^{n+6} \begin{bmatrix} \bar{X}_j^{(c)} \\ \bar{Y}_j^{(c)} \\ \bar{Z}_j^{(c)} \end{bmatrix} q_j \tag{74}$$

where

$$\begin{bmatrix} \bar{X}_j^{(c)} \\ \bar{Y}_j^{(c)} \\ \bar{Z}_j^{(c)} \end{bmatrix} = \sum \begin{bmatrix} \bar{e}_j \\ \bar{f}_j \\ \bar{g}_j \end{bmatrix} . \tag{75}$$

The overall moments about the constant-velocity axes produced by the same force distribution are* (*cf* equations (7) and (65))

$$\begin{aligned} \begin{bmatrix} \bar{L}_c^{(c)} \\ \bar{M}_c^{(c)} \\ \bar{N}_c^{(c)} \end{bmatrix} &= \sum A_{x_c^{(c)}} \begin{bmatrix} \bar{e}^{(c)} \\ \bar{f}^{(c)} \\ \bar{g}^{(c)} \end{bmatrix} = - \sum A_{\bar{e}^{(c)}} \begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} \\ &\approx - \sum A_{\bar{e}_f} \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} - \sum A_{\bar{e}_f} [R \quad I \quad -A_{x_f}] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \\ &\quad + \sum A_{x_f} \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots\cdots\cdots & \\ \bar{g}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \\ &= \text{(say)} \begin{bmatrix} \bar{L}_f \\ \bar{M}_f \\ \bar{N}_f \end{bmatrix} + \sum_{j=1}^{n+6} \begin{bmatrix} \bar{L}_j^{(c)} \\ \bar{M}_j^{(c)} \\ \bar{N}_j^{(c)} \end{bmatrix} q_j \end{aligned} \tag{76}$$

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* A subscript c is required with the symbol for the overall moments but not for the overall forces since only the former depends on the position as well as the orientation of the constant-velocity axes.

$$\text{where} \quad \begin{bmatrix} \bar{L}_1^{(c)} & \cdots & \bar{L}_{n+6}^{(c)} \\ \bar{M}_1^{(c)} & \cdots\cdots\cdots & \\ \bar{N}_1^{(c)} & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} -\sum A_{\bar{z}_f} R & -A_{\bar{x}_f} & \sum A_{\bar{z}_f} A_{x_f} \end{bmatrix}$$

$$+ \sum A_{x_f} \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots\cdots\cdots & \\ \bar{g}_1 & \cdots\cdots\cdots & \end{bmatrix} . \qquad (77)$$

At this point it is worth noting that the perturbation in the moments is not just the moments produced by the perturbations in the local forces, that is not just the second term in the above expression. It should also be clearly understood that the moment we are considering is the moment about the origin of the constant-velocity axes and not about the reference point - hence the use of the subscript c in the symbols for the resolutes of this moment (eg (76)).

Thus finally we have

$$[\bar{Q}_{ij}] = \begin{bmatrix} \sum R^T \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots\cdots\cdots & \\ \bar{g}_1 & \cdots\cdots\cdots & \end{bmatrix} \\ \begin{bmatrix} \bar{X}_1^{(c)} & \cdots & \bar{X}_{n+6}^{(c)} \\ \bar{Y}_1^{(c)} & \cdots\cdots\cdots & \\ \bar{Z}_1^{(c)} & \cdots\cdots\cdots & \end{bmatrix} \\ \begin{bmatrix} \bar{L}_1^{(c)} & \cdots & \bar{L}_{n+6}^{(c)} \\ \bar{M}_1^{(c)} & \cdots\cdots\cdots & \\ \bar{N}_1^{(c)} & \cdots\cdots\cdots & \end{bmatrix} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \sum R^T A_{\bar{z}_f} \\ 0 & 0 & 0 \\ 0 & A_{\bar{X}_f} & \begin{bmatrix} 0 & -\bar{N}_f & \bar{M}_f \\ 0 & 0 & -\bar{L}_f \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} . \qquad (78)$$

Equations (50) (51) and (78) along with (48) provide the means for writing out in full the equations of motion for small perturbations from the datum motion. These are given in Table 3. The various contributions, aerodynamic etc, to the generalised coefficients, have been obtained using the local load vectors given in sections 5.2 to 5.5 (equations (38), (39), (45) and (46)). One may be surprised that the expression for the propulsive matrix $[P_{ij}]$ given in Table 3 involves the constant matrix P_q (see equation (11)). This is a consequence of our assumption that the propulsive force acting on any particular particle has constant components in the direction of the body-fixed axes.

6.2 Equations of motion using body-fixed axes

Initially we will derive the equations of motion, using body-fixed axes, from Lagranges equation for a non-inertial frame (equation (25)) and the equations based on the principle of momentum (equations (26) and (27)) given in section 4. The perturbed position of a particle relative to the body-fixed axes is that given in equation (10), and for the body freedom coordinates we take the translations $\{\hat{x}_1^{(c)}, \hat{y}_1^{(c)}, \hat{z}_1^{(c)}\}$ and the rotations $(\hat{\phi},\hat{\theta},\hat{\psi})$ (*cf* equations (14) and (15)). With this representation the deformational freedoms are deformations relative to the body-fixed axes, which we will call encastré modes. Thus the initial derivation can be described as for:

(i) encastré modes and displacement body freedoms.

In the equations thus derived we can make coordinate transformations which will give different meanings to the generalised coordinates, and so from the initial equations we derive the equations of motion in terms of:

(ii) encastré modes and velocity body freedoms;

(iii) free-free modes and displacement body freedoms; and

(iv) free-free modes and velocity body freedoms.

6.2.1 Equations of motion in terms of encastré modes and displacement body freedoms

The determination of what we call the ponderous terms, those not resulting from any force* but from the fact that a system having mass is moving, has been considered in some detail in Ref 1 (Part II). It is there shown that the ponderous dampings and stiffnesses will all be zero if the angular velocity of the aircraft is zero throughout the datum motion. Indeed, for our chosen datum

* One does of course speak of such things as inertia forces, centrifugal forces, Coriolis forces but these are not really forces but merely a convenient analogy arising from D'Alembert's principle.

motion, in the non-inertial form of Lagrange's equation (see Ref 1, Part II or Ref 15),

$$\frac{\partial V_0}{\partial \hat{q}_i} + J_i + G_i + \frac{d}{dt}\left(\frac{\partial \hat{W}}{\partial \dot{\hat{q}}_i}\right) - \frac{\partial \hat{W}}{\partial \hat{q}_i} = \bar{\hat{Q}}_i \tag{79}$$

there are no first order terms, in the generalised coordinates, arising from G_i or $\partial\hat{W}/\partial\hat{q}_i$.

The linear velocity of the reference point and the angular velocity of the body-fixed axes are (*cf* Ref 1, Part II section 6.4) given by

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \hat{S}\begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} - \hat{S}A_{\hat{x}_1^{(c)}}\hat{S}^T Q_{\hat{\phi}}\begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\psi}} \end{bmatrix} + \hat{S}\begin{bmatrix} \dot{\hat{x}}_1^{(c)} \\ \dot{\hat{y}}_1^{(c)} \\ \dot{\hat{z}}_1^{(c)} \end{bmatrix} - A_p\hat{S}\begin{bmatrix} \hat{x}_1^{(c)} \\ \hat{y}_1^{(c)} \\ \hat{z}_1^{(c)} \end{bmatrix} \tag{80}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = Q_{\hat{\phi}}\begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\psi}} \end{bmatrix} \quad . \tag{81}$$

Thus, with a particle position relative to reference point defined by equation (10), the velocity of the particle is given by

$$\begin{bmatrix} u_m \\ v_m \\ w_m \end{bmatrix} \approx \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + (R - R_0 + A_{x_f}P_q)\begin{bmatrix} \dot{\hat{q}}_1 \\ \vdots \\ \dot{\hat{q}}_n \end{bmatrix} + A_{u_f}\begin{bmatrix} \hat{\phi} \\ \hat{\theta} \\ \hat{\psi} \end{bmatrix} - A_{x_f}\begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\psi}} \end{bmatrix} + \begin{bmatrix} \dot{\hat{x}}_1^{(c)} \\ \dot{\hat{y}}_1^{(c)} \\ \dot{\hat{z}}_1^{(c)} \end{bmatrix} \quad . \tag{82}$$

To the same order of approximation the velocities of the reference frame (equations (80) and (81)) are

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \approx \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + \begin{bmatrix} \dot{\hat{x}}_1^{(c)} \\ \dot{\hat{y}}_1^{(c)} \\ \dot{\hat{z}}_1^{(c)} \end{bmatrix} + A_{u_f} \begin{bmatrix} \hat{\phi} \\ \hat{\theta} \\ \hat{\psi} \end{bmatrix} \tag{83}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} \approx \begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\psi}} \end{bmatrix} . \tag{84}$$

The centrifugal potential function V_0 is therefore for small perturbations, given by

$$\begin{aligned} V_0 &\approx -\tfrac{1}{2}[\dot{\hat{\phi}}\ \dot{\hat{\theta}}\ \dot{\hat{\psi}}]\, I_n \begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\psi}} \end{bmatrix} + \sum \delta m\, [\dot{\hat{\phi}}\ \dot{\hat{\theta}}\ \dot{\hat{\psi}}]\, A_{u_f} (R - R_0 + A_{xf} P_q) \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{bmatrix} \\ &\quad + \sum \delta m\, [\hat{q}_1 \ \dots \ \hat{q}_n] (R^T - R_0^T - P_q^T A_{xf}) \left(A_{u_f} \begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\psi}} \end{bmatrix} + \begin{bmatrix} \ddot{\hat{x}}_1^{(c)} \\ \ddot{\hat{y}}_1^{(c)} \\ \ddot{\hat{z}}_1^{(c)} \end{bmatrix} \right) \\ &= -\tfrac{1}{2}[\dot{\hat{\phi}}\ \dot{\hat{\theta}}\ \dot{\hat{\psi}}]\, I_n \begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\psi}} \end{bmatrix} + [\hat{q}_1 \ \dots \ \hat{q}_n] \left(\left(\sum \delta m R^T \right) - m R_0^T \right) \begin{bmatrix} \ddot{\hat{x}}_1^{(c)} \\ \ddot{\hat{y}}_1^{(c)} \\ \ddot{\hat{z}}_1^{(c)} \end{bmatrix} \end{aligned} \tag{85}$$

and so

$$\begin{bmatrix} \dfrac{\partial V_0}{\partial \hat{q}_1} \\ \vdots \\ \dfrac{\partial V_0}{\partial \hat{q}_n} \end{bmatrix} \approx \left(\left(\sum \delta m R^T \right) - m R_0^T \right) \begin{bmatrix} \ddot{\hat{x}}_1^{(c)} \\ \ddot{\hat{y}}_1^{(c)} \\ \ddot{\hat{z}}_1^{(c)} \end{bmatrix} . \tag{86}$$

The J_i terms are given by

$$\begin{bmatrix} J_1 \\ \vdots \\ J_n \end{bmatrix} \approx -\sum\delta m(R^T - R_0^T - P_q^T A_{x_f})A_{x_f}\begin{bmatrix} \ddot{\hat{\phi}} \\ \ddot{\hat{\theta}} \\ \ddot{\hat{\psi}} \end{bmatrix}$$

$$= \left(-\left(\sum\delta m R^T A_{x_f}\right) - P_q^T I_n\right)\begin{bmatrix} \ddot{\hat{\phi}} \\ \ddot{\hat{\theta}} \\ \ddot{\hat{\psi}} \end{bmatrix} . \quad (87)$$

The kinetic energy relative to the frame of reference (the body-fixed axes) is

$$\hat{W} = \tfrac{1}{2}\sum\delta m[\dot{\hat{q}}_1 \ \dots \ \dot{\hat{q}}_n](R^T - R_0^T - P_q^T A_{x_f})(R - R_0 + A_{x_f}P_q)\begin{bmatrix} \dot{\hat{q}}_1 \\ \vdots \\ \dot{\hat{q}}_n \end{bmatrix} \quad (88)$$

and so

$$\frac{d}{dt}\begin{bmatrix} \dfrac{\partial \hat{W}}{\partial \dot{\hat{q}}_1} \\ \vdots \\ \dfrac{\partial \hat{W}}{\partial \dot{\hat{q}}_n} \end{bmatrix} = [I \ \ -R_0^T \ \ -P_q^T]\begin{bmatrix} \sum\delta m R^T R & \sum\delta m R^T & -\sum\delta m R^T A_{x_f} \\ \sum\delta m R & mI & 0 \\ \sum\delta m A_{x_f} R & 0 & I_n \end{bmatrix}\begin{bmatrix} I \\ -R_0 \\ -P_q \end{bmatrix} . \quad (89)$$

The above equations (86), (87) and (89) suffice to give the ponderous inertia coefficients $\hat{A}_{ij}$ for $i = 1 \rightarrow n$ and all j (see equations (14) and (15)). Details are given in Table 4.

The equations based on the principle momentum are, from (26), (27) and (82)

$$\begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} = \left(\left(\sum\delta m R\right) - mR_0\right)\begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_n \end{bmatrix} + m\begin{bmatrix} \ddot{\hat{x}}_1^{(c)} \\ \ddot{\hat{y}}_1^{(c)} \\ \ddot{\hat{z}}_1^{(c)} \end{bmatrix} \quad (90)$$

$$\begin{bmatrix} \bar{L} \\ \bar{M} \\ \bar{N} \end{bmatrix} = \left(\left(\sum \delta m A_{x_f} R\right) - I_n P_q\right) \begin{bmatrix} \ddot{\hat{q}}_1 \\ \vdots \\ \ddot{\hat{q}}_n \end{bmatrix} + I_n \begin{bmatrix} \ddot{\hat{\phi}} \\ \ddot{\hat{\theta}} \\ \ddot{\hat{\psi}} \end{bmatrix} \tag{91}$$

and so the expressions on the right of these immediately give the other ponderous inertia coefficients (*cf* Table 4).

Due to the force vector

$$\begin{bmatrix} \bar{e} \\ \bar{f} \\ \bar{g} \end{bmatrix} = \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \sum_{i=1}^{n+6} \begin{bmatrix} \bar{\hat{e}}_i \\ \bar{\hat{f}}_i \\ \bar{\hat{g}}_i \end{bmatrix} \hat{q}_i \tag{92}$$

it is easily seen that the translational force and moment about the body-fixed axes are, respectively

$$\begin{aligned} \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} &= \sum \left(\begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \begin{bmatrix} \bar{\hat{e}}_1 & \cdots & \bar{\hat{e}}_{n+6} \\ \bar{\hat{f}}_1 & \cdots\cdots\cdots & \\ \bar{\hat{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \right) \\ &= \text{(say)} \begin{bmatrix} \bar{X}_f \\ \bar{Y}_f \\ \bar{Z}_f \end{bmatrix} + \begin{bmatrix} \bar{\hat{X}}_1 & \cdots & \bar{\hat{X}}_{n+6} \\ \bar{\hat{Y}}_1 & \cdots\cdots\cdots & \\ \bar{\hat{Z}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \end{aligned} \tag{93}$$

and

$$\begin{bmatrix} \bar{L} \\ \bar{M} \\ \bar{N} \end{bmatrix} = \sum A_x \begin{bmatrix} \bar{e} \\ \bar{f} \\ \bar{g} \end{bmatrix}$$

$$\approx \sum \left(A_{x_f} \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + A_{x_f} \begin{bmatrix} \bar{\hat{e}}_1 & \cdots & \bar{\hat{e}}_{n+6} \\ \bar{\hat{f}}_1 & \cdots\cdots\cdots & \\ \bar{\hat{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} - A_{\bar{e}_f} (R - R_0 + A_{x_f} P_q) \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{bmatrix} \right)$$

$$= \text{(say)} \begin{bmatrix} \bar{L}_f \\ \bar{M}_f \\ \bar{N}_f \end{bmatrix} + \begin{bmatrix} \bar{\hat{L}}_1 & \cdots & \bar{\hat{L}}_{n+6} \\ \bar{\hat{M}}_1 & \cdots\cdots\cdots & \\ \bar{\hat{N}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} . \qquad (94)$$

Details of the separate contributions (aerodynamic etc) to these forces, obtained using the different local force vectors (equations (37), (40), (47) and section 5.4), are given in Table 9. As pointed out in section 5.5 and Appendix A, there will be no structural contribution.

For the deformational freedoms the generalised forces resulting from the local force vectors (92), are, using (10) and remembering that the frame of reference is regarded as stationary in the assumed virtual displacement used to calculate the virtual work,

$$\bar{\hat{Q}}_i = (\bar{\hat{Q}}_i)_f + \sum_{j=1}^{n+6} \bar{\hat{Q}}_{ij} \hat{q}_j \qquad (95)$$

where

$$\begin{bmatrix} \bar{\hat{Q}}_{1j} \\ \vdots \\ \bar{\hat{Q}}_{nj} \end{bmatrix} = \sum (R^T - R_0^T - P_q^T A_{x_f}) \begin{bmatrix} \bar{\hat{e}}_j \\ \bar{\hat{f}}_j \\ \bar{\hat{g}}_j \end{bmatrix} . \qquad (96)$$

The individual contributions (aerodynamics etc) to these generalised forces, obtained using the different local force vectors (equations (37), (40), (47) and section 5.4), are given in detail in Table 4.

We have therefore obtained the complete set of equations of motion for small perturbations of the datum motion. The ponderous terms come from equations (86), (87), (89), (90) and (91), and the various forces from equations (93), (94) and (96). Table 4 gives the equations written out in full.

It may be remarked that anybody coming to use the expressions of Table 4 may well not know the individual matrices R, R_0, P_q but only the complete modal matrix $(R - R_0 + A_{x_f}P_q)$ (*cf* equation (10)). The constant matrices R_0 and P_q are, however, in a sense arbitrary. They have only been introduced to ease the understanding of the relationship between one form of the equations of motion and another. Thus if constant terms are added to R_0 and P_q, and corresponding changes made to R so that the new matrix $(R - R_0 + A_{x_f}P_q)$ is exactly the same as the original one, the new matrix R will still satisfy the condition (3). To set up the equations of motion using Table 4 therefore one might as well take P_q and R_0 to be zero, but it must be remembered that R must then, of course, be a matrix of encastré modes.

6.2.2 Equations of motion in terms of encastré modes and velocity body freedoms

The linear velocity of the reference point and the angular velocity of the body-fixed axes are given by equations (80) and (81). Taking the perturbation of these components from their datum motion value as new generalised coordinates for the body freedoms we have

$$\begin{bmatrix} \hat{q}_{n+1} \\ \hat{q}_{n+2} \\ \hat{q}_{n+3} \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} - \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix}$$

$$= (\hat{S} - I)\begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} - \hat{S}A_{\hat{x}_1^{(f)}}\hat{S}^T Q_{\hat{\phi}} \begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\psi}} \end{bmatrix} + \hat{S}\begin{bmatrix} \dot{\hat{x}}_1^{(c)} \\ \dot{\hat{y}}_1^{(c)} \\ \dot{\hat{z}}_1^{(c)} \end{bmatrix} - A_p\hat{S}\begin{bmatrix} \hat{x}_1^{(c)} \\ \hat{y}_1^{(c)} \\ \hat{z}_1^{(c)} \end{bmatrix}$$

$$\approx \begin{bmatrix} \dot{\hat{q}}_{n+1} \\ \dot{\hat{q}}_{n+2} \\ \dot{\hat{q}}_{n+3} \end{bmatrix} + A_{u_f}\begin{bmatrix} \hat{q}_{n+4} \\ \hat{q}_{n+5} \\ \hat{q}_{n+6} \end{bmatrix} , \qquad (97)$$

and

$$\begin{bmatrix} \hat{q}_{n+4} \\ \hat{q}_{n+5} \\ \hat{q}_{n+6} \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = Q_{\hat{\phi}} \begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\psi}} \end{bmatrix} \approx \begin{bmatrix} \dot{\hat{q}}_{n+4} \\ \dot{\hat{q}}_{n+5} \\ \dot{\hat{q}}_{n+6} \end{bmatrix} . \tag{98}$$

The generalised coordinates for the deformational freedoms are kept the same as those used in section 6.2.1 and so all together we have the coordinate transformation

$$\begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \approx \begin{bmatrix} I & 0 & 0 \\ 0 & ID & A_{u_f} \\ 0 & 0 & ID \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \tag{99}$$

which can be inverted to give

$$\begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \approx \begin{bmatrix} I & 0 & 0 \\ 0 & ID^{-1} & -A_{u_f}D^{-2} \\ 0 & 0 & ID^{-1} \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \tag{100}$$

since both sets of coordinates will be zero in the unperturbed state.

One could start afresh and obtain the equations of motion, as in section 6.2.1, using the new coordinates. It is obvious however that the same result is achieved if the transformation (100) is applied to the equation of motion for the case (encastré modes, displacement body freedoms) of section 6.2.1 and Table 4. This has been done with the outcome shown (however see below) in detail in Table 5. It will be seen that the ponderous inertias now give rise to ponderous dampings $\hat{J}_{ij}$ and ponderous stiffnesses $\hat{V}_{ij}$. One would similarly expect terms in D^{-1} and D^{-2} from the various generalised force matrices. However because of the form of these matrices, and in particular the form of the aerodynamic matrix for flight well away from the ground, the only negative power of the differential operator D is that associated with the gravitational force (see Table 5).

It may have been considered desirable to have a symmetric inertia matrix. However this is not possible - to make it symmetric one finds that one has to
077 premultiply the equations of motion by a singular matrix and so would just throw

away some of our information. The submatrix which is the inertia matrix for the degrees of freedom $1 \to n$ (the deformational freedom) will however be symmetric.

In Table 5 some new symbols are used resulting from writing the local aerodynamic force vector (equation (37)) in terms of the new generalised coordinates (equation (99)), ie

$$\begin{bmatrix} e \\ f \\ g \end{bmatrix} \approx \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} + \sum_{j=1}^{n} \begin{bmatrix} \hat{e}_j \\ \hat{f}_j \\ \hat{g}_j \end{bmatrix} \hat{q}_j + \sum_{j=n+1}^{n+6} \begin{bmatrix} \hat{\hat{e}}_j \\ \hat{\hat{f}}_j \\ \hat{\hat{g}}_j \end{bmatrix} \hat{\hat{q}}_j \qquad (101)$$

where

$$\begin{bmatrix} \hat{\hat{e}}_{n+i} \\ \hat{\hat{f}}_{n+i} \\ \hat{\hat{g}}_{n+i} \end{bmatrix} = \begin{bmatrix} \hat{e}_{\dot{n+i}} \\ \hat{f}_{\dot{n+i}} \\ \hat{g}_{\dot{n+i}} \end{bmatrix} \qquad i = 1 \ldots 6 \,. \qquad (102)$$

In the use of Table 5 one can, as pointed out at the end of section 6.2.1, take the matrices P_q and R_0 to be zero and R to be a matrix of encastré modes. The form given there is consequent upon the consistent use of R throughout the paper for a matrix of free-free modes.

6.2.3 Equations of motion in terms of free-free modes and displacement body freedoms

As already pointed out, the use of the non-inertial form of Lagrange's equation (equation (79)) necessitates the choice of coordinates for the deformational degrees of freedom such that the position and orientation of the reference frame (the body-fixed axes in our case) are independent of these coordinates. However one may often prefer to have deformational freedoms which involve some movement of the body-fixed axes, as, for example, when one desires to use normal modes as some of the degrees of freedom. This difficulty is simply overcome by applying a coordinate transformation to the equation of motion obtained in section 6.2.1.

If we put

$$\begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_{n+6} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ -R_0 & I & 0 \\ -P_q & 0 & I \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \qquad (103)$$

ie

$$\begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ R_0 & I & 0 \\ P_q & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_{n+6} \end{bmatrix} \tag{104}$$

then the total perturbation can be thought of as made up of the successive perturbations:

(i) Translations, in the direction of the constant-velocity axes, as a rigid body:

$$\begin{bmatrix} \hat{x}_1^{(c)} \\ \hat{y}_1^{(c)} \\ \hat{z}_1^{(c)} \end{bmatrix} = [R_0 \;\; I \;\; 0] \begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_{n+6} \end{bmatrix} . \tag{105}$$

(ii) Rotations as a rigid body, according to the standard Euler procedure:

$$\begin{bmatrix} \hat{\phi} \\ \hat{\theta} \\ \hat{\psi} \end{bmatrix} = [P_q \;\; 0 \;\; I] \begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_{n+6} \end{bmatrix} . \tag{106}$$

(iii) Deformations:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} = [R - R_0 + A_{x_f} P_q \;\;\; 0 \;\;\; 0] \begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_{n+6} \end{bmatrix} . \tag{107}$$

The position of a particle relative to the origin of the constant-velocity axes and referred to those axes is therefore, from (12), for small perturbations:

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} \approx \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + [R \;\;\; I \;\;\; -A_{x_f}] \begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_{n+6} \end{bmatrix} . \tag{108}$$

Each of the generalised coordinates $\tilde{q}_1 \rightarrow \tilde{q}_n$ is a measure of perturbation just in an unconstrained, or free-free mode - there are no restrictions* on R as distinct from $(R - R_0 + A_{x_f}P_q)$.

As well as making the coordinate transformation (103) we premultiply the equation of motion by the matrix

$$\begin{bmatrix} I & R_0^T & P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{109}$$

so that the symmetry of the ponderous inertia matrix will be preserved. The resultant equation of motion is given in detail in Table 6.

The local aerodynamic force vector, in this case, is written as (*cf* equation (37))

$$\begin{bmatrix} e \\ f \\ g \end{bmatrix} \approx \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} + \sum_{j=1}^{n} \begin{bmatrix} \tilde{e}_j \\ \tilde{f}_j \\ \tilde{g}_j \end{bmatrix} \tilde{q}_j + \begin{bmatrix} \hat{e}_{\dot{n+1}} & \cdots & \hat{e}_{\dot{n+3}} \\ \hat{f}_{\dot{n+1}} & \cdots\cdots\cdots & \\ \hat{g}_{\dot{n+1}} & \cdots\cdots\cdots & \end{bmatrix} \left\{ \begin{bmatrix} \dot{\tilde{q}}_{n+1} \\ \dot{\tilde{q}}_{n+2} \\ \dot{\tilde{q}}_{n+3} \end{bmatrix} + A_{u_f} \begin{bmatrix} \tilde{q}_{n+4} \\ \tilde{q}_{n+5} \\ \tilde{q}_{n+6} \end{bmatrix} \right\}$$

$$+ \begin{bmatrix} \hat{e}_{\dot{n+4}} & \cdots & \hat{e}_{\dot{n+6}} \\ \hat{f}_{\dot{n+4}} & \cdots\cdots\cdots & \\ \hat{g}_{\dot{n+4}} & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} \dot{\tilde{q}}_{n+4} \\ \dot{\tilde{q}}_{n+5} \\ \dot{\tilde{q}}_{n+6} \end{bmatrix} \tag{110}$$

where the aircraft is well away from the ground.

6.2.4 Equations of motion in terms of free-free modes and velocity body freedoms

To change from displacement to velocity generalised coordinates we made the transformation (99). To change from encastré modes to free-free modes we made the transformation (103). To make both these changes we do the latter followed by the former and so we put

* Apart of course from the general restriction that we have imposed throughout that it is such that the body-fixed axes always remain an orthogonal frame.

$$\begin{bmatrix} \check{q}_1 \\ \vdots \\ \check{q}_{n+6} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & ID & A_{u_f} \\ 0 & 0 & ID \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -R_0 & I & 0 \\ -P_q & 0 & I \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \tag{111}$$

ie

$$\begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ R_0 & I & 0 \\ P_q & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & ID^{-1} & -A_{u_f}D^{-2} \\ 0 & 0 & ID^{-1} \end{bmatrix} \begin{bmatrix} \check{q}_1 \\ \vdots \\ \check{q}_{n+6} \end{bmatrix} . \tag{112}$$

The linear and angular velocities of the body-fixed axes are therefore, from (97) and (98)

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \approx \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + [R_0 D + A_{u_f} P_q \quad I \quad 0] \begin{bmatrix} \check{q}_1 \\ \vdots \\ \check{q}_{n+6} \end{bmatrix} \tag{113}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} \approx [P_q D \quad 0 \quad I] \begin{bmatrix} \check{q}_1 \\ \vdots \\ \check{q}_{n+6} \end{bmatrix} . \tag{114}$$

Thus the coordinates $\check{q}_{n+1} \rightarrow \check{q}_{n+6}$ are the components of the linear and angular velocities of the body-fixed axes relative to their values due to the deformation in the free-free modes.

The position of a particle relative to the origin of the constant-velocity axes and referred to those axes is from (12), for small perturbations:

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} \approx \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + \left[R \quad ID^{-1} \quad -(A_{x_f}D^{-1} + A_{u_f}D^{-2}) \right] \begin{bmatrix} \check{q}_1 \\ \vdots \\ \check{q}_{n+6} \end{bmatrix} \tag{115}$$

whereas with the encastré modes of section 6.2.2 it was, from equations (12) and
077 (100), it was:-

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} \approx \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + \left[R - R_0 + A_{x_f} P_q \quad ID^{-1} \quad -(A_{x_f} D^{-1} + A_{u_f} D^{-2}) \right] \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} . \tag{116}$$

In the present case therefore the generalised coordinates $\breve{q}_1 \rightarrow \breve{q}_n$ measure perturbations in unconstrained (free-free) modes, in contrast to the 'encastré modes' which were constrained to have zero displacement and slope at the reference point.

It is desirable that, for the deformational freedoms alone, if not for all the freedoms, the inertia matrix should be symmetric. This is so for all the previous developments and we arrange it to be so now by premultiplying the equations of motion by

$$\begin{bmatrix} I & R_0^T & P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{117}$$

in addition to the coordinate transformation (112). The resultant equation of motion is given in detail in Table 7.

In terms of the current coordinates the local aerodynamic force vector is written as (see equation (37))

$$\begin{bmatrix} e \\ f \\ g \end{bmatrix} \approx \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} + \sum_{j=1}^{n+6} \begin{bmatrix} \breve{e}_j \\ \breve{f}_j \\ \breve{g}_j \end{bmatrix} \breve{q}_j \tag{118}$$

where

$$\begin{bmatrix} \breve{e}_{n+i} \\ \breve{f}_{n+i} \\ \breve{g}_{n+i} \end{bmatrix} = \begin{bmatrix} \hat{e}_{n+i} \\ \hat{f}_{n+i} \\ \hat{g}_{n+i} \end{bmatrix} \qquad i = 1 \ldots 6 \tag{119}$$

the aircraft being well away from the ground.

7 RELATIONSHIPS

7.1 Between local force vectors

In the derivation using constant-velocity axes we have written a typical force vector (referred to the same axes) as (equation (65))

$$\begin{bmatrix} \bar{e}^{(c)} \\ \bar{f}^{(c)} \\ \bar{g}^{(c)} \end{bmatrix} \approx \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots\cdots\cdots & \\ \bar{g}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \tag{120}$$

while in the body-fixed axes derivation the typical force vector is variously written as* (equation (92))

$$\begin{aligned} \begin{bmatrix} \bar{e} \\ \bar{f} \\ \bar{g} \end{bmatrix} &\approx \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \begin{bmatrix} \bar{\hat{e}}_1 & \cdots & \bar{\hat{e}}_{n+6} \\ \bar{\hat{f}}_1 & \cdots\cdots\cdots & \\ \bar{\hat{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \\ &= \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \begin{bmatrix} \bar{\overset{\frown}{e}}_1 & \cdots & \bar{\overset{\frown}{e}}_{n+6} \\ \bar{\overset{\frown}{f}}_1 & \cdots\cdots\cdots & \\ \bar{\overset{\frown}{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} \overset{\frown}{q}_1 \\ \vdots \\ \overset{\frown}{q}_{n+6} \end{bmatrix} \\ &= \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \begin{bmatrix} \bar{\tilde{e}}_1 & \cdots & \bar{\tilde{e}}_{n+6} \\ \bar{\tilde{f}}_1 & \cdots\cdots\cdots & \\ \bar{\tilde{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_{n+6} \end{bmatrix} \\ &= \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \begin{bmatrix} \bar{\breve{e}}_1 & \cdots & \bar{\breve{e}}_{n+6} \\ \bar{\breve{f}}_1 & \cdots\cdots\cdots & \\ \bar{\breve{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} \breve{q}_1 \\ \vdots \\ \breve{q}_{n+6} \end{bmatrix} . \end{aligned} \tag{121}$$

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* The dressings $\hat{}$, $\overset{\frown}{}$, $\tilde{}$, $\breve{}$ refer respectively to the cases encastré modes-displacement body freedoms, encastré-velocity, free-free-displacement, and free-free-velocity.

Referring the latter to the constant-velocity (ie datum-attitude earth) axes we have

$$\begin{bmatrix} \bar{e}^{(c)} \\ \bar{f}^{(c)} \\ \bar{g}^{(c)} \end{bmatrix} \approx \begin{bmatrix} \bar{e} \\ \bar{f} \\ \bar{g} \end{bmatrix} - A_{\bar{e}_f} \begin{bmatrix} \hat{q}_{n+4} \\ \hat{q}_{n+5} \\ \hat{q}_{n+6} \end{bmatrix} \tag{122}$$

where, from equations (98), (104) and (112)

$$\begin{bmatrix} \hat{q}_{n+4} \\ \hat{q}_{n+5} \\ \hat{q}_{n+6} \end{bmatrix} \approx D^{-1} \begin{bmatrix} \hat{q}_{n+4} \\ \hat{q}_{n+5} \\ \hat{q}_{n+6} \end{bmatrix} \approx [P_q \quad 0 \quad I] \begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_{n+6} \end{bmatrix}$$

$$\approx [P_q \quad 0 \quad ID^{-1}] \begin{bmatrix} \check{q}_1 \\ \vdots \\ \check{q}_{n+6} \end{bmatrix} . \tag{123}$$

If the typical particle position vectors, relative to the origin of the constant-velocity axes and referred to the same axes are, to first order of smallness, the same in each derivation then the force vectors will be the same, to first order, in each derivation. Thus if (*cf* equations (7) and (13))

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} \approx \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + [R \quad I \quad -A_{x_f}] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix}$$

$$\approx \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + [R - R_0 + A_{x_f}P_q \quad I \quad -A_{x_f}] \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \tag{124}$$

then the corresponding force vectors will be equal for small perturbations. Equation (124) is satisfied if

$$\begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ R_0 & I & 0 \\ P_q & 0 & I \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \tag{125}$$

and so from (120) and (122)

$$\begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots\cdots\cdots & \\ \bar{g}_1 & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} \bar{\hat{e}}_1 & \cdots & \bar{\hat{e}}_{n+6} \\ \bar{\hat{f}}_1 & \cdots\cdots\cdots & \\ \bar{\hat{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ R_0 & I & 0 \\ P_q & 0 & I \end{bmatrix} - A_{e_f}^{-} [P_q \ 0 \ I] \quad . \tag{126}$$

For the other sets of generalised coordinates (124) is satisfied by (see equations (99), (103) and (111))

$$\begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ R_0 D + A_{u_f} P_q & ID & A_{u_f} \\ P_q D & 0 & ID \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \tag{127}$$

or

$$\begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_{n+6} \end{bmatrix} = \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \tag{128}$$

or

$$\begin{bmatrix} \check{q}_1 \\ \vdots \\ \check{q}_{n+6} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & ID & A_{u_f} \\ 0 & 0 & ID \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \quad . \tag{129}$$

Consequently we find that

$$\begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots\cdots\cdots & \\ \bar{g}_1 & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} \bar{\widehat{e}}_1 & \cdots & \bar{\widehat{e}}_{n+6} \\ \bar{\widehat{f}}_1 & \cdots\cdots\cdots & \\ \bar{\widehat{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ R_0D + A_{u_f}P_q & ID & A_{u_f} \\ P_qD & 0 & ID \end{bmatrix} - A_{\bar{e}_f} [P_q \;\; 0 \;\; I]$$

$$= \begin{bmatrix} \bar{\tilde{e}}_1 & \cdots & \bar{\tilde{e}}_{n+6} \\ \bar{\tilde{f}}_1 & \cdots\cdots\cdots & \\ \bar{\tilde{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} - A_{\bar{e}_f} [P_q \;\; 0 \;\; I]$$

$$= \begin{bmatrix} \bar{\check{e}}_1 & \cdots & \bar{\check{e}}_{n+6} \\ \bar{\check{f}}_1 & \cdots\cdots\cdots & \\ \bar{\check{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & ID & A_{u_f} \\ 0 & 0 & ID \end{bmatrix} - A_{\bar{e}_f} [P_q \;\; 0 \;\; I] \quad . \qquad (130)$$

Equations (126) and (130) are the required relationships between the various forms of the local force vector. Putting them another way they are

$$\begin{bmatrix} \bar{\hat{e}}_1 & \cdots & \bar{\hat{e}}_{n+6} \\ \bar{\hat{f}}_1 & \cdots\cdots\cdots & \\ \bar{\hat{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots\cdots\cdots & \\ \bar{g}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -R_0 & I & 0 \\ -P_q & 0 & I \end{bmatrix} + A_{\bar{e}_f} [0 \;\; 0 \;\; I]$$

$$= \begin{bmatrix} \bar{\widehat{e}}_1 & \cdots & \bar{\widehat{e}}_{n+6} \\ \bar{\widehat{f}}_1 & \cdots\cdots\cdots & \\ \bar{\widehat{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & ID & A_{u_f} \\ 0 & 0 & ID \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\tilde{e}}_1 & \cdots & \bar{\tilde{e}}_{n+6} \\ \bar{\tilde{f}}_1 & \cdots\cdots\cdots & \\ \bar{\tilde{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -R_0 & I & 0 \\ -P_q & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\check{e}}_1 & \cdots & \bar{\check{e}}_{n+6} \\ \bar{\check{f}}_1 & \cdots\cdots\cdots & \\ \bar{\check{g}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -(R_0D + A_{u_f}P_q) & ID & A_{u_f} \\ -P_qD & 0 & ID \end{bmatrix} \quad . \qquad (131)$$

7.1.1 Between local aerodynamic force vectors when well away from the ground

For an aircraft in flight well away from the ground, as in the assumption throughout the main part of this paper, it can reasonably be assumed that an adequate approximation to the local aerodynamic force vector has the form given in equation (37) (for body-fixed axes, encastré modes and displacement body freedoms) or (38) (for constant-velocity axes). That is, with expressions of the forms given in equations (120) or (121), certain coefficients have the following character:

$$\begin{bmatrix} \hat{e}_{n+1} & \cdots & \hat{e}_{n+3} \\ \hat{f}_{n+1} & \cdots\cdots\cdots & \\ \hat{g}_{n+1} & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{e}_{n\dot{+}1} & \cdots & \hat{e}_{n\dot{+}3} \\ \hat{f}_{n\dot{+}1} & \cdots\cdots\cdots & \\ \hat{g}_{n\dot{+}1} & \cdots\cdots\cdots & \end{bmatrix} D \tag{132}$$

$$\begin{bmatrix} \hat{e}_{n+4} & \cdots & \hat{e}_{n+6} \\ \hat{f}_{n+4} & \cdots\cdots\cdots & \\ \hat{g}_{n+4} & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{e}_{n\dot{+}1} & \cdots & \hat{e}_{n\dot{+}3} \\ \hat{f}_{n\dot{+}1} & \cdots\cdots\cdots & \\ \hat{g}_{n\dot{+}1} & \cdots\cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} \hat{e}_{n\dot{+}4} & \cdots & \hat{e}_{n\dot{+}6} \\ \hat{f}_{n\dot{+}4} & \cdots\cdots\cdots & \\ \hat{g}_{n\dot{+}4} & \cdots\cdots\cdots & \end{bmatrix} D \tag{133}$$

$$\begin{bmatrix} e_{n+1} & \cdots & e_{n+3} \\ f_{n+1} & \cdots\cdots\cdots & \\ g_{n+1} & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} e_{n\dot{+}1} & \cdots & e_{n\dot{+}3} \\ f_{n\dot{+}1} & \cdots\cdots\cdots & \\ g_{n\dot{+}1} & \cdots\cdots\cdots & \end{bmatrix} \tag{134}$$

$$\begin{bmatrix} e_{n+4} & \cdots & e_{n+6} \\ f_{n+4} & \cdots\cdots\cdots & \\ g_{n+4} & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} e_{n\dot{+}1} & \cdots & e_{n\dot{+}3} \\ f_{n\dot{+}1} & \cdots\cdots\cdots & \\ g_{n\dot{+}1} & \cdots\cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} e_{n\dot{+}4} & \cdots & e_{n\dot{+}6} \\ f_{n\dot{+}4} & \cdots\cdots\cdots & \\ g_{n\dot{+}4} & \cdots\cdots\cdots & \end{bmatrix} D - A_{e_f} \;. \tag{135}$$

It follows from (126) that

$$\begin{bmatrix} e_1 & \cdots & e_n \\ f_1 & \cdots\cdots & \\ g_1 & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{e}_1 & \cdots & \hat{e}_n \\ \hat{f}_1 & \cdots\cdots & \\ \hat{g}_1 & \cdots\cdots & \end{bmatrix} - A_{e_f}P_q$$

$$+ \begin{bmatrix} \hat{e}_{\dot{n+1}} & \cdots & \hat{e}_{\dot{n+3}} \\ \hat{f}_{\dot{n+1}} & \cdots\cdots & \\ \hat{g}_{\dot{n+1}} & \cdots\cdots & \end{bmatrix} (R_0 D + A_{u_f}P_q) + \begin{bmatrix} \hat{e}_{\dot{n+4}} & \cdots & \hat{e}_{\dot{n+6}} \\ \hat{f}_{\dot{n+4}} & \cdots\cdots & \\ \hat{g}_{\dot{n+4}} & \cdots\cdots & \end{bmatrix} P_q D \qquad \ldots\ldots (136)$$

and

$$\begin{bmatrix} e_{\dot{n+i}} \\ f_{\dot{n+i}} \\ g_{\dot{n+i}} \end{bmatrix} = \begin{bmatrix} \hat{e}_{\dot{n+i}} \\ \hat{f}_{\dot{n+i}} \\ \hat{g}_{\dot{n+i}} \end{bmatrix} \qquad i = 1 \rightarrow 6 \ . \ (137)$$

With the other sets of generalised coordinates we have (*cf* equations (101), (102), (*110*), (*118*) and (*119*))

$$\begin{bmatrix} \tilde{e}_1 & \cdots & \tilde{e}_n \\ \tilde{f}_1 & \cdots\cdots & \\ \tilde{g}_1 & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \breve{e}_1 & \cdots & \breve{e}_n \\ \breve{f}_1 & \cdots\cdots & \\ \breve{g}_1 & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{e}_1 & \cdots & \hat{e}_n \\ \hat{f}_1 & \cdots\cdots & \\ \hat{g}_1 & \cdots\cdots & \end{bmatrix}$$

$$+ \begin{bmatrix} \hat{e}_{\dot{n+1}} & \cdots & \hat{e}_{\dot{n+3}} \\ \hat{f}_{\dot{n+1}} & \cdots\cdots & \\ \hat{g}_{\dot{n+1}} & \cdots\cdots & \end{bmatrix} (R_0 D + A_{u_f}P_q) + \begin{bmatrix} \hat{e}_{\dot{n+4}} & \cdots & \hat{e}_{\dot{n+6}} \\ \hat{f}_{\dot{n+4}} & \cdots\cdots & \\ \hat{g}_{\dot{n+4}} & \cdots\cdots & \end{bmatrix} P_q D \qquad \ldots\ldots (138)$$

$$\begin{bmatrix} \hat{e}_1 & \cdots & \hat{e}_n \\ \hat{f}_1 & \cdots\cdots & \\ \hat{g}_1 & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{e}_1 & \cdots & \hat{e}_n \\ \hat{f}_1 & \cdots\cdots & \\ \hat{g}_1 & \cdots\cdots & \end{bmatrix} \qquad (139)$$

$$\begin{bmatrix} \hat{e}_{n+i} \\ \hat{f}_{n+i} \\ \hat{g}_{n+i} \end{bmatrix} = \begin{bmatrix} \check{e}_{n+i} \\ \check{f}_{n+i} \\ \check{g}_{n+i} \end{bmatrix} = \begin{bmatrix} \hat{e}_{\dot{n+i}} \\ \hat{f}_{\dot{n+i}} \\ \hat{g}_{\dot{n+i}} \end{bmatrix} \qquad i = 1 \ldots 6 \tag{140}$$

$$\begin{bmatrix} \tilde{e}_{n+1} & \ldots & \tilde{e}_{n+3} \\ \tilde{f}_{n+1} & \ldots\ldots\ldots \\ \tilde{g}_{n+1} & \ldots\ldots\ldots \end{bmatrix} = \begin{bmatrix} \hat{e}_{\dot{n+1}} & \ldots & \hat{e}_{\dot{n+3}} \\ \hat{f}_{\dot{n+1}} & \ldots\ldots\ldots \\ \hat{g}_{\dot{n+1}} & \ldots\ldots\ldots \end{bmatrix} D \tag{141}$$

and

$$\begin{bmatrix} \tilde{e}_{n+4} & \ldots & \tilde{e}_{n+6} \\ \tilde{f}_{n+4} & \ldots\ldots\ldots \\ \tilde{g}_{n+4} & \ldots\ldots\ldots \end{bmatrix} = \begin{bmatrix} \hat{e}_{\dot{n+1}} & \ldots & \hat{e}_{\dot{n+3}} \\ \hat{f}_{\dot{n+1}} & \ldots\ldots\ldots \\ \hat{g}_{\dot{n+1}} & \ldots\ldots\ldots \end{bmatrix} A_{u_f} + \begin{bmatrix} \hat{e}_{\dot{n+4}} & \ldots & \hat{e}_{\dot{n+6}} \\ \hat{f}_{\dot{n+4}} & \ldots\ldots\ldots \\ \hat{g}_{\dot{n+4}} & \ldots\ldots\ldots \end{bmatrix} D \,. \tag{142}$$

It will be noticed that the relationships (139), (141) and (142) have already been substituted in equations (101) and (110), as appropriate, and the symbols $\hat{e}_i$ $(i = 1 \rightarrow n)$ and $\tilde{e}_{n+i}$ $(i = 1 \rightarrow 6)$ are not used outside this section. Equation (138) gives the relationship one would expect between the two cases where one has free-free deformation modes (the $\sim$ and the $\smile$); (139) what is expected for the two cases with encastré modes (the $\hat{}$ and the $\frown$); (140) what is expected for the two cases with velocity rigid body coordinates (the $\frown$ and the $\smile$); and (141) what is expected for the cases with displacement rigid body coordinates (the $\hat{}$ and the $\sim$). Finally it is interesting to note that, from (136) and (138),

$$\begin{bmatrix} \tilde{e}_1 & \ldots & \tilde{e}_n \\ \tilde{f}_1 & \ldots\ldots \\ \tilde{g}_1 & \ldots\ldots \end{bmatrix} = \begin{bmatrix} \check{e}_1 & \ldots & \check{e}_n \\ \check{f}_1 & \ldots\ldots \\ \check{g}_1 & \ldots\ldots \end{bmatrix} = \begin{bmatrix} e_1 & \ldots & e_n \\ f_1 & \ldots\ldots \\ g_1 & \ldots\ldots \end{bmatrix} + A_{e_f} P_q \,. \tag{143}$$

This relationship could, of course, have been immediately obtained from (130). Equations (139) and (143) are in fact independent of the particular form of the aerodynamic forces consequent upon remoteness from the ground.

7.2 Between overall force and moment vector coefficients

From the expressions for the overall forces and moments (equations (74), (76), (93) and (94)) and the relationship (131) between the local force vectors it is easily seen that we have the following connection between the coefficients for the constant-velocity axes development and those for the basic body-fixed axes (encastré modes displacement body freedoms) development:

$$\begin{bmatrix} \bar{\hat{X}}_1 & \cdots & \bar{\hat{X}}_{n+6} \\ \bar{\hat{Y}}_1 & \cdots\cdots\cdots & \\ \bar{\hat{Z}}_1 & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} \bar{X}_1^{(c)} & \cdots & \bar{X}_{n+6}^{(c)} \\ \bar{Y}_1^{(c)} & \cdots\cdots\cdots & \\ \bar{Z}_1^{(c)} & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -R_0 & I & 0 \\ -P_q & 0 & I \end{bmatrix} + [0 \quad 0 \quad A_{\bar{X}_f}] \tag{144}$$

$$\begin{bmatrix} \bar{\hat{L}}_1 & \cdots & \bar{\hat{L}}_{n+6} \\ \bar{\hat{M}}_1 & \cdots\cdots\cdots & \\ \bar{\hat{N}}_1 & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} \bar{L}_1^{(c)} & \cdots & \bar{L}_{n+6}^{(c)} \\ \bar{M}_1^{(c)} & \cdots\cdots\cdots & \\ \bar{N}_1^{(c)} & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -R_0 & I & 0 \\ -P_q & 0 & I \end{bmatrix} + [0 \quad A_{\bar{X}_f} \quad A_{\bar{L}_f}]. \tag{145}$$

Using the relationships between the various sets of body-fixed generalised coordinates (equations (99), (103) and (111)), we also immediately find that:

$$\begin{bmatrix} \bar{\hat{X}}_1 & \cdots & \bar{\hat{X}}_{n+6} \\ \bar{\hat{Y}}_1 & \cdots\cdots\cdots & \\ \bar{\hat{Z}}_1 & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} \bar{\hat{\hat{X}}}_1 & \cdots & \bar{\hat{\hat{X}}}_{n+6} \\ \bar{\hat{\hat{Y}}}_1 & \cdots\cdots\cdots & \\ \bar{\hat{\hat{Z}}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & ID & Au_f \\ 0 & 0 & ID \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\tilde{X}}_1 & \cdots & \bar{\tilde{X}}_{n+6} \\ \bar{\tilde{Y}}_1 & \cdots\cdots\cdots & \\ \bar{\tilde{Z}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -R_0 & I & 0 \\ -P_q & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\breve{X}}_1 & \cdots & \bar{\breve{X}}_{n+6} \\ \bar{\breve{Y}}_1 & \cdots\cdots\cdots & \\ \bar{\breve{Z}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -(R_0 D + Au_f P_q) & ID & Au_f \\ -P_q D & 0 & ID \end{bmatrix} \tag{146}$$

and

$$\begin{bmatrix} \bar{\hat{L}}_1 & \cdots & \bar{\hat{L}}_{n+6} \\ \bar{\hat{M}}_1 & \cdots\cdots\cdots & \\ \bar{\hat{N}}_1 & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} \bar{\hat{L}}_1 & \cdots & \bar{\hat{L}}_{n+6} \\ \bar{\hat{M}}_1 & \cdots\cdots\cdots & \\ \bar{\hat{N}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & ID & A_{u_f} \\ 0 & 0 & ID \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\tilde{L}}_1 & \cdots & \bar{\tilde{L}}_{n+6} \\ \bar{\tilde{M}}_1 & \cdots\cdots\cdots & \\ \bar{\tilde{N}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -R_0 & I & 0 \\ -P_q & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\check{L}}_1 & \cdots & \bar{\check{L}}_{n+6} \\ \bar{\check{M}}_1 & \cdots\cdots\cdots & \\ \bar{\check{N}}_1 & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -(R_0 D + A_{u_f} P_q) & ID & A_{u_f} \\ -P_q D & 0 & ID \end{bmatrix} . \qquad (147)$$

7.2.1 *Between overall aerodynamic force and moment vector coefficients when well away from the ground*

With the particular forms of the local aerodynamic force vectors appropriate to flight where there is no ground effect (equations (37), (38), (101), (110) and (118)) it follows that certain coefficients have particular forms (*cf* Tables 8 to 12)

$$\begin{bmatrix} X^{(c)}_{n+1} & \cdots & X^{(c)}_{n+3} \\ Y^{(c)}_{n+1} & \cdots\cdots\cdots & \\ Z^{(c)}_{n+1} & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} X^{(c)}_{\dot{x}} & \cdots & X^{(c)}_{\dot{z}} \\ Y_{\dot{x}} & \cdots\cdots\cdots & \\ Z_{\dot{x}} & \cdots\cdots\cdots & \end{bmatrix} D \qquad (148)$$

$$\begin{bmatrix} X^{(c)}_{n+4} & \cdots & X^{(c)}_{n+6} \\ Y^{(c)}_{n+4} & \cdots\cdots\cdots & \\ Z^{(c)}_{n+4} & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} X^{(c)}_{\dot{x}} & \cdots & X^{(c)}_{\dot{z}} \\ Y^{(c)}_{\dot{x}} & \cdots\cdots\cdots & \\ Z^{(c)}_{\dot{x}} & \cdots\cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} X^{(c)}_{\dot{\phi}} & \cdots & X^{(c)}_{\dot{\psi}} \\ Y^{(c)}_{\dot{\phi}} & \cdots\cdots\cdots & \\ Z^{(c)}_{\dot{\phi}} & \cdots\cdots\cdots & \end{bmatrix} D - A_{X_f}$$

$$\cdots\cdots\cdots \quad (149)$$

$$\begin{bmatrix} L_{n+1}^{(c)} & \cdots & L_{n+3}^{(c)} \\ M_{n+1}^{(c)} & \cdots\cdots & \\ N_{n+1}^{(c)} & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} L_{\dot{x}}^{(c)} & \cdots & L_{\dot{z}}^{(c)} \\ M_{\dot{x}}^{(c)} & \cdots\cdots & \\ N_{\dot{x}}^{(c)} & \cdots\cdots & \end{bmatrix} D - A_{X_f} \tag{150}$$

$$\begin{bmatrix} L_{n+4}^{(c)} & \cdots & L_{n+6}^{(c)} \\ M_{n+4}^{(c)} & \cdots\cdots & \\ N_{n+4}^{(c)} & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} L_{\dot{x}}^{(c)} & \cdots & L_{\dot{z}}^{(c)} \\ M_{\dot{x}}^{(c)} & \cdots\cdots & \\ N_{\dot{x}}^{(c)} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} L_{\dot{\phi}}^{(c)} & \cdots & L_{\dot{\psi}}^{(c)} \\ M_{\dot{\phi}}^{(c)} & \cdots\cdots & \\ N_{\dot{\phi}}^{(c)} & \cdots\cdots & \end{bmatrix} D - A_{L_f} \tag{151}$$

and similarly for the $\hat{}$ and $\tilde{}$ coefficients except that the term A_{X_f}, A_{L_f} are absent. For the $\overset{\frown}{}$ coefficients (and similarly for the $\check{}$ coefficients) we can simply write $\overset{\frown}{X}_{n+1} = \overset{\frown}{X}_u$, $\overset{\frown}{X}_{n+4} = \overset{\frown}{X}_p$ etc. It is then easily seen (*cf* section 7.2) that

$$\left.\begin{aligned} X_{\dot{x}}^{(c)} &= \hat{X}_{\dot{x}} = \tilde{X}_{\dot{x}} = \overset{\frown}{X}_u = \check{X}_u \\ X_{\dot{\phi}}^{(c)} &= \hat{X}_{\dot{\phi}} = \tilde{X}_{\dot{\phi}} = \overset{\frown}{X}_p = \check{X}_p \\ L_{\dot{x}}^{(c)} &= \hat{L}_{\dot{x}} = \tilde{L}_{\dot{x}} = \overset{\frown}{L}_u = \check{L}_u \\ L_{\dot{\phi}}^{(c)} &= \hat{L}_{\dot{\phi}} = \tilde{L}_{\dot{\phi}} = \overset{\frown}{L}_p = \check{L}_p \quad \text{etc.} \end{aligned}\right\} \tag{152}$$

$$\begin{aligned} \begin{bmatrix} \hat{X}_1 & \cdots & \hat{X}_n \\ \hat{Y}_1 & \cdots\cdots & \\ \hat{Z}_1 & \cdots\cdots & \end{bmatrix} &= \begin{bmatrix} X_1^{(c)} & \cdots & X_n^{(c)} \\ Y_1^{(c)} & \cdots\cdots & \\ Z_1^{(c)} & \cdots\cdots & \end{bmatrix} - \begin{bmatrix} X_{\dot{x}}^{(c)} & \cdots & X_{\dot{z}}^{(c)} \\ Y_{\dot{x}}^{(c)} & \cdots\cdots & \\ Z_{\dot{x}}^{(c)} & \cdots\cdots & \end{bmatrix} (R_0 D + A_{u_f} P_q) \\ &\quad - \begin{bmatrix} X_{\dot{\phi}}^{(c)} & \cdots & X_{\dot{\psi}}^{(c)} \\ Y_{\dot{\phi}}^{(c)} & \cdots\cdots & \\ Z_{\dot{\phi}}^{(c)} & \cdots\cdots & \end{bmatrix} P_q D + A_{X_f} P_q \\ &= \begin{bmatrix} \overset{\frown}{X}_1 & \cdots & \overset{\frown}{X}_n \\ \overset{\frown}{Y}_1 & \cdots\cdots & \\ \overset{\frown}{Z}_1 & \cdots\cdots & \end{bmatrix}, \end{aligned} \tag{153}$$

$$\begin{bmatrix} \hat{L}_1 & \cdots & \hat{L}_n \\ \hat{M}_1 & \cdots\cdots & \\ \hat{N}_1 & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} L_1^{(c)} & \cdots & L_n^{(c)} \\ M_1^{(c)} & \cdots\cdots & \\ N_1^{(c)} & \cdots\cdots & \end{bmatrix} - \begin{bmatrix} L_{\dot{x}}^{(c)} & \cdots & L_{\dot{z}}^{(c)} \\ M_{\dot{x}}^{(c)} & \cdots\cdots & \\ N_{\dot{x}}^{(c)} & \cdots\cdots & \end{bmatrix} (R_0 D + A_{u_f} P_q)$$

$$- \begin{bmatrix} L_{\dot{\phi}}^{(c)} & \cdots & L_{\dot{\psi}}^{(c)} \\ M_{\dot{\phi}}^{(c)} & \cdots\cdots & \\ N_{\dot{\phi}}^{(c)} & \cdots\cdots & \end{bmatrix} P_q D + A_{X_f} R_0 + A_{L_f} P_q$$

$$= \begin{bmatrix} \hat{L}_1 & \cdots & \hat{L}_n \\ \hat{M}_1 & \cdots\cdots & \\ \hat{N}_1 & \cdots\cdots & \end{bmatrix} , \tag{154}$$

$$\begin{bmatrix} X_1^{(c)} & \cdots & X_n^{(c)} \\ Y_1^{(c)} & \cdots\cdots & \\ Z_1^{(c)} & \cdots\cdots & \end{bmatrix} + A_{X_f} P_q = \begin{bmatrix} \tilde{X}_1 & \cdots & \tilde{X}_n \\ \tilde{Y}_1 & \cdots\cdots & \\ \tilde{Z}_1 & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \check{X}_1 & \cdots & \check{X}_n \\ \check{Y}_1 & \cdots\cdots & \\ \check{Z}_1 & \cdots\cdots & \end{bmatrix} \tag{155}$$

$$\begin{bmatrix} L_1^{(c)} & \cdots & L_n^{(c)} \\ M_1^{(c)} & \cdots\cdots & \\ N_1^{(c)} & \cdots\cdots & \end{bmatrix} + A_{X_f} R_0 + A_{L_f} P_q = \begin{bmatrix} \tilde{L}_1 & \cdots & \tilde{L}_n \\ \tilde{M}_1 & \cdots\cdots & \\ \tilde{N}_1 & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \check{L}_1 & \cdots & \check{L}_n \\ \check{M}_1 & \cdots\cdots & \\ \check{N}_1 & \cdots\cdots & \end{bmatrix} . \quad \ldots\ldots \tag{156}$$

The above relationships can be confirmed from the expression of Tables 8 to 12 using the relationships between the local aerodynamic force vectors given in section 4.1.1. Anybody with prior knowledge of relationships between aerodynamic derivatives may well be surprised by the equations (152). They are used to relationships such as (see *eg* Ref 2, section 22), for our case of zero angular velocity during the datum motion,

$$Z_{\dot{\phi}} = Z_p + w_f Z_{\dot{v}} - v_f Z_{\dot{w}} \quad . \tag{157}$$

It must be emphasised therefore that our terms $\hat{Z}_{\dot{\phi}}$ are not derivatives. In general they are differential operators. Thus we could write (the notation is ephemeral)

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$$\left.\begin{aligned} \hat{Z}_{\dot{\phi}} &= z_{\dot{\phi}} + z_{\ddot{\phi}}D + \dots \\ \hat{\hat{Z}}_p &= z_p + z_{\dot{p}}D + \dots \end{aligned}\right\} \tag{158}$$

and then, making use of (97) and (98), we find that the coefficients of $\dot{\phi}$ in $\hat{Z}$ and $\hat{\hat{Z}}$ are respectively

$$\left.\begin{aligned} &z_{\dot{\phi}} + z_{\ddot{y}}w_f - z_{\ddot{z}}v_f \\ \text{and} \quad &z_p + z_{\dot{v}}w_f - z_{\dot{w}}v_f \quad . \end{aligned}\right\} \tag{159}$$

The former term is commonly called $Z_{\dot{\phi}}$ since it has come from a Taylor expansion in terms of the displacements; while the latter term is written as the linear combination $(Z_p + Z_{\dot{v}}w_f - Z_{\dot{w}}v_f)$ of three derivatives since a Taylor expansion in terms of velocities was first used before transforming to the displacement coordinates. Equating these two expressions then gives (157) which contrasts with our relationship

$$\hat{Z}_{\dot{\phi}} = \hat{\hat{Z}}_p \tag{160}$$

where the quantities involved are not derivatives.

7.3 Between coefficients in perturbation equations of motion

In the constant-velocity axes derivation a general expression $[\bar{Q}_{ij}]$ for any of the matrices, $[Q_{ij}]$, $-[E_{ij}]$, $-[P_{ij}]$, $-[G_{ij}]$ was obtained in equation (78). The corresponding general matrix $[\hat{Q}_{ij}]$ for the body-fixed axes derivation using encastré modes and displacement body freedoms is given by equations (93), (94) and (96). From the relationships between the local force vectors (equation (131)) and between the overall forces (equations (144) and (145)) we then find that

$$[\bar{\hat{Q}}_{ij}] = \begin{bmatrix} I & -R_0^T & -P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} [\bar{Q}_{ij}] \begin{bmatrix} I & 0 & 0 \\ -R_0 & I & 0 \\ -P_q & 0 & I \end{bmatrix}$$

$$+ \begin{bmatrix} I & -R_0^T & -P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \left(\sum R^T A_{\bar{e}_f}\right) P_q & 0 & 0 \\ 0 & 0 & A_{\bar{X}_f} \\ A_{\bar{X}_f} R_0 + \begin{bmatrix} 0 & -\bar{N}_f & \bar{M}_f \\ 0 & 0 & -\bar{L}_f \\ 0 & 0 & 0 \end{bmatrix} P_q & 0 & \begin{bmatrix} 0 & 0 & 0 \\ \bar{N}_f & 0 & 0 \\ -\bar{M}_f & \bar{L}_f & 0 \end{bmatrix} \end{bmatrix}$$

$$+ \begin{bmatrix} P_q^T \left\{ A_{\bar{X}_f} R_0 - \sum A_{\bar{e}_f} R - \left(\sum A_{\bar{e}_f} A_{x_f}\right) P_q \right\} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \qquad (161)$$

In addition we have the relationship between the ponderous inertia matrices (*cf* Tables 3 and 4)

$$[\hat{A}_{ij}] = \begin{bmatrix} I & -R_0^T & -P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} [A_{ij}] \begin{bmatrix} I & 0 & 0 \\ -R_0 & I & 0 \\ -P_q & 0 & I \end{bmatrix} . \qquad (162)$$

The three other forms of the perturbation equation of motion that we have considered were all obtained simply from the equation for the body-fixed axes, encastré modes, displacement body freedoms, case by a coordinate transformation and possibly also a matrix premultiplication (see sections 6.2.2, 6.2.3 and 6.2.4). Thus we have

$$[\bar{Q}_{ij}] = [\bar{\hat{Q}}_{ij}] \begin{bmatrix} I & 0 & 0 \\ 0 & ID^{-1} & -A_{u_f} D^{-2} \\ 0 & 0 & ID^{-1} \end{bmatrix} \qquad (163)$$

$$[\bar{\tilde{Q}}_{ij}] = \begin{bmatrix} I & R_0^T & P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} [\bar{\hat{Q}}_{ij}] \begin{bmatrix} I & 0 & 0 \\ R_0 & I & 0 \\ P_q & 0 & I \end{bmatrix} \tag{164}$$

$$[\bar{\breve{Q}}_{ij}] = \begin{bmatrix} I & R_0^T & P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} [\bar{\hat{Q}}_{ij}] \begin{bmatrix} I & 0 & 0 \\ R_0 & ID^{-1} & -A_{u_f}D^{-2} \\ P_q & 0 & ID^{-1} \end{bmatrix} \tag{165}$$

where each of these matrices may be a Q_{ij}, $-E_{ij}$, $-P_{ij}$, or $-G_{ij}$ (or $-G_{ij} - G^*_{ij}D^{-1}$) matrix with the appropriate dressing. Similarly for the ponderous matrices we have

$$[\hat{A}_{ij}] = [\hat{A}_{ij}] \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{166}$$

$$[\hat{J}_{ij}] = [\hat{A}_{ij}] \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{167}$$

$$[\hat{V}_{ij}] = [\hat{A}_{ij}] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -A_{u_f} \\ 0 & 0 & 0 \end{bmatrix} \tag{168}$$

$$[\tilde{A}_{ij}] = [A_{ij}] \tag{169}$$

$$[\breve{A}_{ij}] = [A_{ij}] \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{170}$$

$$[\breve{J}_{ij}] = [A_{ij}] \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{171}$$

$$[\check{V}_{ij}] = [A_{ij}] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -A_{u_f} \\ 0 & 0 & 0 \end{bmatrix} \quad . \tag{172}$$

The various quantities required, in addition to the coefficients in the original form of the perturbation equation of motion, to transform to another form of the equation are summarised in Table 13. The transformations between the various forms derived using body-fixed axes, require, when there is a change between encastré and free-free modes, some modal information in the form of the matrices R_0, P_q . In addition certain information on the datum-motion forces is required when transforming to or from the constant-velocity axes form. For example $\sum A_{\bar{e}_f} A_{x_f}$ is required with $A_{\bar{e}_f}$ replaced in turn by A_{e_f}, $A_{e_{pf}}$, $A_{e_{sf}}$ and $A_{e_{gf}}$ ($\equiv g\delta m A_{\ell\Phi_f}$). The sum of the forces on any particle during the datum motion will be zero* and so, for example

$$\sum\left(A_{e_f} + A_{e_{pf}} + A_{e_{sf}} + A_{e_{gf}}\right)A_{x_f} = 0 \tag{173}$$

though

$$\sum A_{e_f} A_{x_f} \neq 0 \quad . \tag{174}$$

One can therefore assume, in transforming to or from the constant-velocity axes form, that all the individual datum motion forces are zero. The result will be an equation which has the right solution though the individual terms are in general wrong, being different from those obtained when the equation is derived directly. Thus, for example, for a rigid aircraft (ie n = 0) in a vacuum, since from the equilibrium equation (Table 1 or 2) we then have $A_{X_{gf}} = -A_{X_{pf}}$, we have (*cf* Tables 3 and 4)

$$[G_{ij}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} , \qquad [P_{ij}] = \begin{bmatrix} 0 & -A_{X_{gf}} \\ 0 & 0 \end{bmatrix} \tag{175}$$

and

$$[\hat{G}_{ij}] = \begin{bmatrix} 0 & -A_{X_{gf}} \\ 0 & 0 \end{bmatrix} , \qquad [\hat{P}_{ij}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{176}$$

*** This, of course, is more than is said by the datum motion equations for it is inconceivable that the degrees of freedom should be as numerous as the particles of the aircraft.**

in the constant-velocity form and body-fixed form (displacement body freedoms) of the perturbation equations respectively. However, if we transform from one to the other, using equation (161), and in the transformation assume the datum motion forces are zero, then we find that these gravitational and propulsive matrices are unchanged.

7.4 With certain other published equations

A number of authors (Milne[6], Etkin[7], Taylor[1], and others) have suggested the use of mean-body axes in the derivation of the equations of motion of a deformable aircraft. In our derivations of sections 6.2.3 and 6.2.4 a frame of reference, occupying the location the body-fixed axes would have if there were no deformation (ie if the first n generalised coordinates were zero), can be thought of as mean-body axes provided the modal matrix R satisfies the conditions* for mean-body axes, viz:

$$\left.\begin{aligned} \sum \delta m R &= 0 \\ \sum \delta m A_{x_f} R &= 0 \quad . \end{aligned}\right\} \qquad (177)$$

The equation given in detail in Table 7 does not agree, for example, as regards individual terms with that obtained by Etkin[7]. Like us he used Lagrange's equation for a non-inertial frame in conjunction with body freedom equations obtained by the principles of momentum. However he naturally considered the components of the momentum referred to the mean-body axes rather than the body-fixed axes. It is easily seen that the mean-body axes become coincident with the body-fixed axes if they are given translations

$$R_0 \begin{bmatrix} \check{q}_1 \\ \vdots \\ \check{q}_n \end{bmatrix} \qquad (178)$$

and rotations

$$P_q \begin{bmatrix} \check{q}_1 \\ \vdots \\ \check{q}_n \end{bmatrix} \qquad (179)$$

* A way, but not the only way, of satisfying (177) is to take as modes a set of the free-free normal modes in the absence of gravity.

(*cf* equations (115), (2) and (11)). If we denote by superscript (m) quantities referred to the mean-body axes* we can write, using (5),

$$\begin{bmatrix} \bar{X}^{(m)} \\ \bar{Y}^{(m)} \\ \bar{Z}^{(m)} \end{bmatrix} \approx \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} - A_{\bar{X}_f} P_q \begin{bmatrix} \check{q}_1 \\ \vdots \\ \check{q}_n \end{bmatrix} \tag{180}$$

and

$$\begin{bmatrix} \bar{L}_m^{(m)} \\ \bar{M}_m^{(m)} \\ \bar{N}_m^{(m)} \end{bmatrix} \approx \begin{bmatrix} \bar{L} \\ \bar{M} \\ \bar{N} \end{bmatrix} - (A_{\bar{X}_f} R_0 + A_{\bar{L}_f} P_q) \begin{bmatrix} \check{q}_1 \\ \vdots \\ \check{q}_n \end{bmatrix} \tag{181}$$

where $\bar{X}$, $\bar{L}$ etc can be written in terms of the $\check{q}_i$ using equations (93), (94) and (112). Thus if we add**

$$\begin{bmatrix} 0 & 0 & 0 \\ A_{X_{gf}} P_q & 0 & 0 \\ A_{X_{gf}} R_0 + A_{L_{gf}} P_q & 0 & 0 \end{bmatrix} \tag{182}$$

to $[\check{G}_{ij}]$, and apply the conditions (177); and similarly with the matrices $[\check{P}_{ij}]$ and $[-\check{Q}_{ij}]$, we should, and indeed do, get the body-freedom equations† given by Etkin[7]. He, incidentally, follows the fairly common practice of including the propulsive terms in the aerodynamic matrix; and he also chooses his modes to satisfy the normal mode condition, that $\sum \delta m R^T R$ is diagonal, in addition to (177).

It also can be shown that our deformation freedom equations of Table 7, subject to the mean-body axes condition (equation (177)) become identical with Etkins[7], in their non-concise form†, if a term (*cf* equations (161) and (163))

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* Thus $\bar{L}_m^{(m)}$, $\bar{M}_m^{(m)}$, $\bar{N}_m^{(m)}$, are typical moment components about the origin of the mean-body axes (hence the subscript) and referred to those axes (hence the superscript).

** In fact $A_{L_{gf}} = 0$; and there also is no change to $[\check{E}_{ij}]$ since the structure cannot exert any overall force on itself.

† In the 'non-concise' form they had before division by an appropriate mass or moment of inertia.

$$P_q^T\left(A_{X_{gf}} R_0 - \sum A_{e_{gf}} R - \left(\sum A_{e_{gf}} A_{x_f}\right) P_q\right) \tag{183}$$

is added to $[\breve{G}_{ij}]$ in the 11 submatrix position, and similar additions are made to $[\breve{P}_{ij}]$, $[\breve{E}_{ij}]$ and $[-\breve{Q}_{ij}]$. Some of these additional terms are of course zero as a consequence of the characteristics of the gravitational and structural forces (sections 5.3 and 5.5) and of the conditions (177).

Summing up, therefore, we see that we can transform, from our perturbation motion equation obtained using body-fixed axes, free-free modes, and velocity body freedoms (Table 7 and section 6.2.3) with deformation modes satisfying the conditions (177), to equations obtained using mean-body axes and similar freedoms by adding

$$\begin{bmatrix} P_q^T\left(A_{X_{gf}} R_0 - \sum A_{e_{gf}} R - \left(\sum A_{e_{gf}} A_{x_f}\right) P_q\right) & 0 & 0 \\ A_{X_{gf}} P_q & 0 & 0 \\ A_{X_{gf}} R_0 + A_{L_{gf}} P_q & 0 & 0 \end{bmatrix} \tag{184}$$

to $[\breve{G}_{ij}]$ and making similar additions to $[\breve{P}_{ij}]$, $[\breve{E}_{ij}]$ and $[-\breve{Q}_{ij}]$. Since, as noted in section 7.3, the sum of the forces on any particle during the datum motion will be zero, it follows that the sum of these additions will be zero. The resultant equations are given in detail in Table 14. Note that the aerodynamic overall force and moment about the origin of the mean-body axes, referred to the mean-body axes, are (*cf* equations (155), (156), (180) and (181))

$$\begin{bmatrix} X^{(m)} \\ Y^{(m)} \\ Z^{(m)} \end{bmatrix} = \left[\begin{bmatrix} X_1^{(c)} & \cdots & X_n^{(c)} \\ Y_1^{(c)} & \cdots\cdots & \\ Z_1^{(c)} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \hat{X}_u & \cdots & \hat{X}_w \\ \hat{Y}_u & \cdots\cdots & \\ \hat{Z}_u & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \hat{X}_p & \cdots & \hat{X}_r \\ \hat{Y}_p & \cdots\cdots & \\ \hat{Z}_p & \cdots\cdots & \end{bmatrix}\right] \begin{bmatrix} \breve{q}_1 \\ \vdots \\ \breve{q}_{n+6} \end{bmatrix} \tag{185}$$

$$\begin{bmatrix} L_m^{(m)} \\ M_m^{(m)} \\ N_m^{(m)} \end{bmatrix} = \left[\begin{bmatrix} L_1^{(c)} & \cdots & L_n^{(c)} \\ M_1^{(c)} & \cdots\cdots & \\ N_1^{(c)} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \hat{L}_u & \cdots & \hat{L}_w \\ \hat{M}_u & \cdots\cdots & \\ \hat{N}_u & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \hat{L}_p & \cdots & \hat{L}_r \\ \hat{M}_p & \cdots\cdots & \\ \hat{N}_p & \cdots\cdots & \end{bmatrix}\right] \begin{bmatrix} \breve{q}_1 \\ \vdots \\ \breve{q}_{n+6} \end{bmatrix} . \tag{186}$$

Though considering the use of various frames of reference - mean-body axes, body-fixed axes, principal axes - Milne (Ref 6, Part I, Appendix 1) states the perturbation equations of motion in a concentrated form appropriate to any type of body axes. Thus his equations, when the perturbations are small, with the body axes chosen to be coincident with the principal axes of inertia through the aircraft centre of gravity during the datum motion (ie equations (52) and (53) being satisfied), are identical with ours for body-fixed axes, free-free modes and velocity body freedoms (Table 7 and section 6.2.3) when the matrices (184) etc are added to our constituent matrices of Table 7.

For a rigid aircraft it is usual to reduce the perturbation equations of motion to what Hopkin[2] called 'a concise form' by dividing each force equation by the mass and each moment equation by the appropriate moment of inertia. Etkin[7] and others (Ref 8 for example) have similarly set up a 'concise' form for the equations of motion of the deformable aircraft. Thus their final form[7,8] is that given when the equation of Table 14 is premultiplied by

$$\begin{bmatrix} \sum \delta m R^T R & 0 & 0 \\ 0 & mI & 0 \\ 0 & 0 & I_n \end{bmatrix}^{-1} . \qquad (187)$$

Finally we would just mention that various other authors (Refs 9 to 13 *et al*) have derived or quoted the perturbation equations of motion for a deformable aircraft. For the datum motion that we have considered it is believed that their forms are all included in the cases we have already considered though the notation and/or assumptions used are not always clear.

8 CONCLUDING REMARKS

We have been able to develop in detail various forms of the perturbation equations of motion of a deformable aircraft and present them in the attached tables (Tables 3 to 7 and 14). These relate to small perturbations from a datum motion which is straight flight with constant linear and zero angular velocity. The relationships between the various forms were established and stated, and the requirements for transformation from one form to another listed (Table 13). A particular feature of the work has been the care with which account has been taken of the various forces (structural etc) which were already present in the unperturbed state. To make use of the equations the reader will of course have to call in the help of the aerodynamicist and the elastician (or make use of

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the results of a ground resonance test as in Appendix C) and possibly others as well.

There is room for further development: different unperturbed conditions, better representation of the propulsive forces, more realistic model of structural damping, more accurate representation of the hereditary nature of the aerodynamic forces, etc - but the need for such refinements has yet to be convincingly demonstrated.

Appendix A

STRUCTURAL FORCES FOR AN ISOTROPIC ELASTIC BODY

The meaning of the term 'structural force' on a particle or element is perhaps better understood when it is defined in terms of stresses for an isotropic elastic body with no structural damping. Then (*cf* eg Ref 3) the structural force is given by, for the unperturbed state

$$\begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} = \begin{bmatrix} \int_{\text{element}} \left(\frac{\partial \sigma_{xx}}{\partial x_f} + \frac{\partial \sigma_{xy}}{\partial y_f} + \frac{\partial \sigma_{xz}}{\partial z_f} \right) dx_f dy_f dz_f \\ \int_{\text{element}} \left(\frac{\partial \sigma_{yx}}{\partial x_f} + \frac{\partial \sigma_{yy}}{\partial y_f} + \frac{\partial \sigma_{yz}}{\partial z_f} \right) dx_f dy_f dz_f \\ \int_{\text{element}} \left(\frac{\partial \sigma_{zx}}{\partial x_f} + \frac{\partial \sigma_{zy}}{\partial y_f} + \frac{\partial \sigma_{zz}}{\partial z_f} \right) dx_f dy_f dz_f \end{bmatrix}$$

$$- \begin{bmatrix} \int_{\text{surface}} (\sigma_{xx}\ell + \sigma_{xy}m + \sigma_{xz}n) d(\text{surface}) \\ \int_{\text{surface}} (\sigma_{yx}\ell + \sigma_{yy}m + \sigma_{yz}n) d(\text{surface}) \\ \int_{\text{surface}} (\sigma_{zx}\ell + \sigma_{zy}m + \sigma_{zz}n) d(\text{surface}) \end{bmatrix} \qquad \text{(A-1)}$$

where σ_{xx} etc are the stress components ($\sigma_{xy} = \sigma_{yx}$ etc)

$\int_{\text{surface}}$ means the integral over the free surface of the element

ℓ, m, n are the direction cosines of the outward drawn normal to the surface.

The first term is the structural force there would be on an entirely internal element of the body, and the second is the reduction consequent from the fact that there is no structure to exert a force on the free surface of an element. It is

immediately obvious, using Green's formula relating surface and volume integrals, that the sum of the above structural forces over the whole body is zero and so

$$\begin{bmatrix} X_{sf} \\ Y_{sf} \\ Z_{sf} \end{bmatrix} = \sum \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} = 0 \quad . \tag{A-2}$$

We also have from Green's formula, with S enclosing the volume V

$$\int_V \left(y_f \frac{\partial \sigma_{xy}}{\partial y_f} - \sigma_{xy} \right) dV = \int_S m y_f \sigma_{xy} dS \tag{A-3}$$

and

$$\int_V \left(x_f \frac{\partial \sigma_{yx}}{\partial x_f} - \sigma_{yx} \right) dV = \int_S \ell x_f \sigma_{yx} dS \quad . \tag{A-4}$$

Consequently, since $\sigma_{xy} = \sigma_{yx}$

$$\int_V \left(y_f \frac{\partial \sigma_{xy}}{\partial y_f} - x_f \frac{\partial \sigma_{yx}}{\partial x_f} \right) dV = \int_S \left(m y_f \sigma_{xy} - \ell x_f \sigma_{yx} \right) dS \tag{A-5}$$

and so

$$\begin{aligned} &\int_V y_f \left(\frac{\partial \sigma_{xx}}{\partial x_f} + \frac{\partial \sigma_{xy}}{\partial y_f} + \frac{\partial \sigma_{xz}}{\partial z_f} \right) dV \\ &\qquad - \int_V x_f \left(\frac{\partial \sigma_{yx}}{\partial z_f} + \frac{\partial \sigma_{yy}}{\partial y_f} + \frac{\partial \sigma_{yz}}{\partial z_f} \right) dV \\ &= \int_S y_f \left(\ell \sigma_{xx} + m \sigma_{xy} + n \sigma_{xz} \right) dS \\ &\qquad - \int_S x_f \left(\ell \sigma_{yx} + m \sigma_{yy} + n \sigma_{yz} \right) dS \quad . \end{aligned} \tag{A-6}$$

Using this and similar relationships derived in the same way we find therefore that

$$\begin{bmatrix} L_{sf} \\ M_{sf} \\ N_{sf} \end{bmatrix} = \sum A_{x_f} \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} = 0 \tag{A-7}$$

where the summation is over the whole body.

In the perturbed state described by equation (4) the structural force on a particle will have the form, referred to the constant velocity axes:

$$\begin{bmatrix} e_s^{(c)} \\ f_s^{(c)} \\ g_s^{(c)} \end{bmatrix} \approx S^T \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} + \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \tag{A-8}$$

since the structural force, referred to the body-fixed axes, will be unchanged for any perturbation without any deformation. The above analysis can therefore be repeated using this force vector and replacing (x_f, y_f, z_f) by $(x_c^{(c)}, y_c^{(c)}, z_c^{(c)})$. Consequently we find that

$$\sum \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} = 0 \tag{A-9}$$

$$\sum A_{x_c^{(c)}} \begin{bmatrix} e_s^{(c)} \\ f_s^{(c)} \\ g_s^{(c)} \end{bmatrix} = 0 \quad . \tag{A-10}$$

Expanding the latter equation in terms of the generalised coordinates (see equations (5), (7) and (A-8)) and using the fact that

$$A_{x_c^{(c)}} \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} = - A_{e_{sf}} \begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} \tag{A-11}$$

we get from the first order terms the relationship

$$\sum \begin{bmatrix} Ax_f \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots\cdots & \\ g_{s1} & \cdots\cdots\cdots & \end{bmatrix} & 0 & -Ax_f Ae_{sf} \end{bmatrix} = \sum Ae_{sf} [R \quad I \quad -Ax_f] \tag{A-12}$$

and so

$$\sum Ax_f \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots\cdots & \\ g_{s1} & \cdots\cdots\cdots & \end{bmatrix} = \sum Ae_{sf} R \tag{A-13}$$

$$\sum (Ax_f Ae_{sf} - Ae_{sf} Ax_f) = 0 \quad . \tag{A-14}$$

The latter equation is of course equivalent to (A-7). Now the general expressions for the overall forces and moments, referred to the constant-velocity axes, (equations (74) and (75)) give, using (A-8) or (46), structural contributions due to the perturbations:

$$\begin{bmatrix} X_s^{(c)} \\ Y_s^{(c)} \\ Z_s^{(c)} \end{bmatrix} - \begin{bmatrix} X_{sf} \\ Y_{sf} \\ Z_{sf} \end{bmatrix} \approx \begin{bmatrix} \sum \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots\cdots & \\ g_{s1} & \cdots\cdots\cdots & \end{bmatrix} & 0 & -\sum Ae_{sf} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \tag{A-15}$$

$$\begin{bmatrix} L_{sc}^{(c)} \\ M_{sc}^{(c)} \\ N_{sc}^{(c)} \end{bmatrix} - \begin{bmatrix} L_{sf} \\ M_{sf} \\ N_{sf} \end{bmatrix} \approx \begin{bmatrix} \sum Ax_f \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} - \sum Ae_{sf} R & -Ax_{sf} & \sum Ae_{sf} Ax_f - \sum Ax_f Ae_{sf} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} .$$

...... (A-16)

Consequently from (A-2), (A-9), (A-13) and (A-14) all these contributions are zero.

The above expressions (A-15) and (A-16) are of course independent of the assumption of an isotropic elastic body; and so if we assert that a structure cannot exert any overall translational force or moment on itself we immediately

find that the relationships (A-2), (A-7), (A-9), (A-13) and (A-14) are true in general.

So far we have taken no account of the relationships between stress and strain and displacement. The strain components are simply related to the first derivatives of the displacements; and for our 'isotropic elastic body' the stress and strain components are linearly related by Hooke's Law. Consequently it can be shown that the strain energy of a deformed elastic body is a quadratic function of the first derivatives of the displacements (*cf* Ref 14). Thus, in our case, the strain energy E is a quadratic function of the generalised coordinates q_i :

$$E = E_0 + [(E_i)_f] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} + \tfrac{1}{2} [q_1 \; \dots \; q_{n+6}] [E_{ij}] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \qquad \text{(A-17)}$$

where, without any loss in generality, $[E_{ij}]$ is written as a symmetric matrix. From the fact of the existence of such a potential function it immediately follows that the matrix of the generalised structural force coefficients, in the *perturbation equation of motion*, is, as we have already indicated by our notation, $[E_{ij}]$ which is symmetric. Thus from the form already deduced for this matrix (Table 3) we see that

$$\sum R^T \begin{bmatrix} e_{s1} & \dots & e_{sn} \\ f_{s1} & \dots\dots\dots & \\ g_{s1} & \dots\dots\dots & \end{bmatrix} \quad \text{is symmetric}$$

ie

$$\sum \begin{bmatrix} e_{s1} & f_{s1} & g_{s1} \\ \vdots & \vdots & \vdots \\ e_{sn} & \vdots & \vdots \end{bmatrix} R = \sum R^T \begin{bmatrix} e_{s1} & \dots & e_{sn} \\ f_{s1} & \dots\dots\dots & \\ g_{s1} & \dots\dots\dots & \end{bmatrix} . \qquad \text{(A-18)}$$

Appendix B

PERTURBATION EQUATIONS OF MOTION NEAR THE GROUND

For an aircraft in flight close to the ground the boundary condition to be satisfied by the air flow at the surface of the ground has an important effect on the air forces applied to the aircraft. Consequently the forms taken, in the main part of this Report, for the local aerodynamic force vector at the surface of the aircraft (equations (37), (38), (101), (110) and (118)) are no longer an adequate representation of the truth. It is necessary then to write the local aerodynamic force vector in the appropriate one of the forms we have taken for the typical local force vector (*cf* for example equations (120) and (121) in section 4.1). The matrix of generalised aerodynamic force coefficients, for the constant-velocity axes derivation, is consequently given by equation (78) with the '-' dressing omitted. Similarly the other forms are obtained using the relationships (161), (163), (164) and (165) - again omitting the bars -, and put in terms of the desired local and overall aerodynamic load coefficients by the use of equations (126), (130), (144), (145), (146) and (147). The resultant aerodynamic matrices are given in detail in Table 15. For the various body-fixed axes cases the local aerodynamic force vector has been written as (*cf* equations (100), (104) and (112))

$$\begin{bmatrix} e \\ f \\ g \end{bmatrix} \approx \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} + \sum_{j=1}^{n+6} \begin{bmatrix} \hat{e}_j \\ \hat{f}_j \\ \hat{g}_j \end{bmatrix} \hat{q}_j$$

$$\approx \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} + \sum_{j=1}^{n} \begin{bmatrix} \hat{e}_j \\ \hat{f}_j \\ \hat{g}_j \end{bmatrix} \hat{q}_j + \sum_{j=n+1}^{n+3} \begin{bmatrix} \hat{e}_j \\ \hat{f}_j \\ \hat{g}_j \end{bmatrix} (D^{-1}\hat{q}_j - Au_f D^{-2}\hat{q}_{j+3})$$

$$+ \sum_{j=n+4}^{n+6} \begin{bmatrix} \hat{e}_j \\ \hat{f}_j \\ \hat{g}_j \end{bmatrix} D^{-1}\hat{q}_j$$

$$\approx \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} + \sum_{j=1}^{n} \begin{bmatrix} \tilde{e}_j \\ \tilde{f}_j \\ \tilde{g}_j \end{bmatrix} \tilde{q}_j + \sum_{j=n+1}^{n+6} \begin{bmatrix} \hat{e}_j \\ \hat{f}_j \\ \hat{g}_j \end{bmatrix} \tilde{q}_j$$

$$\approx \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} + \sum_{j=1}^{n} \begin{bmatrix} \tilde{e}_j \\ \tilde{f}_j \\ \tilde{g}_j \end{bmatrix} \check{q}_j + \sum_{j=n+1}^{n+3} \begin{bmatrix} \hat{e}_j \\ \hat{f}_j \\ \hat{g}_j \end{bmatrix} (D^{-1}\check{q}_j - A u_f D^{-2}\check{q}_{j+3}) + \sum_{j=n+4}^{n+6} \begin{bmatrix} \hat{e}_j \\ \hat{f}_j \\ \hat{g}_j \end{bmatrix} D^{-1}\check{q}_j$$

...... (B-1)

where

$$\begin{bmatrix} \tilde{e}_1 & \cdots & \tilde{e}_n \\ \tilde{f}_1 & \cdots\cdots & \\ \tilde{g}_1 & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{e}_1 & \cdots & \hat{e}_{n+6} \\ \hat{f}_1 & \cdots\cdots & \\ \hat{g}_1 & \cdots\cdots & \end{bmatrix} \begin{bmatrix} I \\ R_0 \\ P_q \end{bmatrix} \qquad \text{(B-2)}$$

and the coefficients $\hat{e}_j$, $\tilde{e}_j$ etc are, in general, differential operators. Similar expressions hold, and have been used, with the subscripts (x,y,z,ϕ,θ,ψ) replacing $(n+1,\ldots,n+6)$, for the overall forces and moments. Comparing (37) with (B-1), and so on, we note that

$$\begin{bmatrix} \hat{e}_{n+1} & \cdots & \hat{e}_{n+6} \\ \hat{f}_{n+1} & \cdots\cdots & \\ \hat{g}_{n+1} & \cdots\cdots & \end{bmatrix} \longrightarrow \begin{bmatrix} \hat{e}_{\dot{n}+1} & \cdots & \hat{e}_{\dot{n}+6} \\ \hat{f}_{\dot{n}+1} & \cdots\cdots & \\ \hat{g}_{\dot{n}+1} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} D & A u_f \\ 0 & D \end{bmatrix} \qquad \text{(B-3)}$$

etc, as the distance of the aircraft from the ground becomes large.

B.1 Upholding forces

When there is contact between the aircraft and the ground, as, for example, during the take-off run, additional forces are obviously felt by the aircraft. These we will call upholding, or support, forces. We will assume the normal earth-fixed axes $0_0x_0y_0z_0$ have been defined such that during the datum motion the aircraft has zero velocity components in the directions 0_0y_0 and 0_0z_0. Its velocity u_0 in the direction 0_0x_0 will be constant, in accord with the

assumption throughout this Report that any datum motion considered is one of zero angular velocity and constant linear velocity. One case of certain particular interest (*cf* for example Appendix C) *is that for which* u_0 *is zero. We also* assume the earth surface, when under no external load, is given by

$$z_0 = z_p^{(0)}(x_0, y_0) \tag{B-4}$$

and that the earth behaves elastically* such that when displaced to $z_0 = z^{(0)}(x_0, y_0)$ it exerts a force

$$-\delta_u^{(0)}(x_0, y_0)\begin{bmatrix} 0 \\ 0 \\ z^{(0)} - z_p^{(0)} \end{bmatrix} \tag{B-5}$$

at $(x_0, y_0, z^{(0)})$, referred to the normal earth-fixed axes.

Transformation from the normal earth-fixed axes to the constant-velocity is accomplished by

(i) translations $(u_0 t, 0, 0)$ *in the direction of the normal earth-fixed* axes;

(ii) rotations $(\Phi_f, \Theta_f, \Psi_f)$ in the usual Euler order about the carried axes.

Thus we are assuming that the reference point coincides with the origin 0_0 of the normal earth-fixed axes at the instance $t = 0$ during the datum motion. Writing

$$S = [j_\phi,\ k_\phi,\ \ell_\phi] \tag{B-6}$$

we find that a particle on the outside of the aircraft, which was at the point (x_f, y_f, z_f) in the unperturbed state, is in contact with the ground at an instant during small perturbations of the datum motion provided that

$$\ell_{\Phi_f}^T \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} - z_p^{(0)} + \left[-\frac{\partial z_p^{(0)}}{\partial x_0},\ -\frac{\partial z_p^{(0)}}{\partial y_0},\ 1 \right] S_{\Phi_f}^T \begin{bmatrix} x_c^{(c)} - x_f \\ y_c^{(c)} - y_f \\ z_c^{(c)} - z_f \end{bmatrix} > 0 \tag{B-7}$$

* If desired a more general representation could be used with $\delta_u^{(0)}$ taken to be a differential operator.

where ℓ_{Φ_f} is given by equation (39), $z_p^{(0)}$ and its derivatives are evaluated at

$$(x_0, y_0) = \left(j_{\Phi_f}^T \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + u_0 t \, , \quad k_{\Phi_f}^T \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} \right) \tag{B-8}$$

and from (B-6),

$$j_{\Phi_f} = \begin{bmatrix} \cos\Theta_f \cos\Psi_f \\ -\cos\Phi_f \sin\Psi_f + \sin\Phi_f \cos\Psi_f \sin\Theta_f \\ \sin\Phi_f \sin\Psi_f + \cos\Phi_f \cos\Psi_f \sin\Theta_f \end{bmatrix} \tag{B-9}$$

$$k_{\Phi_f} = \begin{bmatrix} \cos\Theta_f \sin\Psi_f \\ \cos\Phi_f \cos\Psi_f + \sin\Phi_f \sin\Psi_f \sin\Theta_f \\ -\sin\Phi_f \cos\Psi_f + \cos\Phi_f \sin\Psi_f \sin\Theta_f \end{bmatrix} . \tag{B-10}$$

Consequently, provided (B-7) is satisfied, the ground exerts a force on that particle of the aircraft which, referred to the constant-velocity axes is given by the vector

$$\begin{bmatrix} e_u^{(c)} \\ f_u^{(c)} \\ g_u^{(c)} \end{bmatrix} \approx -\delta_u^{(0)} \ell_{\Phi_f} \left\{ \left(\ell_{\Phi_f}^T \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} - z_p^{(0)} \right) \right.$$

$$+ \left(\left(\ell_{\Phi_f}^T \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} - z_p^{(0)} \right) \begin{bmatrix} \dfrac{\partial \delta_u^{(0)}}{\partial x_0} \, , & \dfrac{\partial \delta_u^{(0)}}{\partial y_0} \, , & 0 \end{bmatrix} \right.$$

$$\left. \left. + \begin{bmatrix} -\dfrac{\partial z_p^{(0)}}{\partial x_0} \, , & -\dfrac{\partial z_p^{(0)}}{\partial y_0} \, , & 1 \end{bmatrix} \right) S_{\Phi_f}^T \begin{bmatrix} x^{(c)} - x_f \\ y^{(c)} - y_f \\ z^{(c)} - z_f \end{bmatrix} \right\}$$

...... (B-11)

where $\delta_u^{(0)}$, $z_p^{(0)}$ and their derivatives are evaluated at the point given by (B-8).

Now during the datum motion these forces must be constant and so we must have either

$$u_0 = 0 \tag{B-12}$$

or

$$\frac{\partial z_p^{(0)}}{\partial x_0} = \frac{\partial \delta_u^{(0)}}{\partial x_0} = 0 \qquad \text{everywhere .} \tag{B-13}$$

With either of these conditions satisfied we can write, at the point given by (B-8),

$$z_p^{(0)} = z_p(x_f, y_f, z_f) \tag{B-14}$$

$$\delta_u^{(0)} = \delta_u(x_f, y_f, z_f) \tag{B-15}$$

and

$$\delta_u^{(0)} \left\{ \ell_{\Phi_f}^T \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} - z_p^{(0)} \right\} = - g_{u_f}^{(0)}(x_f, y_f, z_f) \quad . \tag{B-16}$$

Thus (B-11) becomes, remembering that S is orthogonal

$$\begin{bmatrix} e_u^{(c)} \\ f_u^{(c)} \\ g_u^{(c)} \end{bmatrix} \approx g_u^{(0)} \ell_{\Phi_f} - \left\{ \delta_u \ell_{\Phi_f} \ell_{\Phi_f}^T - g_{u_f}^{(0)} \ell_{\Phi_f} \left[\frac{\partial \delta_u}{\partial x_f} , \frac{\partial \delta_u}{\partial y_f} , \frac{\partial \delta_u}{\partial z_f} \right] - \delta_u \ell_{\Phi_f} \left[\frac{\partial z_p}{\partial x_f} , \frac{\partial z_p}{\partial y_f} , \frac{\partial z_p}{\partial z_f} \right] \right\} \begin{bmatrix} x^{(c)} - x_f \\ y^{(c)} - y_f \\ z^{(c)} - z_f \end{bmatrix} . \tag{B-17}$$

Note that, from (B-14),

$$\begin{bmatrix} \partial z_p / \partial x_f \\ \partial z_p / \partial y_f \\ \partial z_p / \partial z_f \end{bmatrix} = \frac{\partial z_p^{(0)}}{\partial x_0} i_{\Phi_f} + \frac{\partial z_p^{(0)}}{\partial y_0} k_{\Phi_f} \tag{B-18}$$

and so

$$S^T_{\Phi_f} \begin{bmatrix} \partial z_p/\partial x_f \\ \partial z_p/\partial y_f \\ \partial z_p/\partial z_f \end{bmatrix} = \begin{bmatrix} \partial z_p^{(0)}/\partial x_0 \\ \partial z_p^{(0)}/\partial y_0 \\ 0 \end{bmatrix} . \quad \text{(B-19)}$$

Similarly

$$S^T_{\Phi_f} \begin{bmatrix} \partial \delta_u/\partial x_f \\ \partial \delta_u/\partial y_f \\ \partial \delta_u/\partial z_f \end{bmatrix} = \begin{bmatrix} \partial \delta_u^{(0)}/\partial x_0 \\ \partial \delta_u^{(0)}/\partial y_0 \\ 0 \end{bmatrix} . \quad \text{(B-20)}$$

Thus the functions δ_u, z_p must be such that

$$\begin{bmatrix} \frac{\partial \delta_u}{\partial x_f} & \frac{\partial \delta_u}{\partial y_f} & \frac{\partial \delta_u}{\partial z_f} \end{bmatrix} \ell_{\Phi_f} = 0 \quad \text{(B-21)}$$

and

$$\begin{bmatrix} \frac{\partial z_p}{\partial x_f} & \frac{\partial z_p}{\partial y_f} & \frac{\partial z_p}{\partial z_f} \end{bmatrix} \ell_{\Phi_f} = 0 . \quad \text{(B-22)}$$

In addition, if the condition (B-13) is the one which is satisfied during the datum motion, then these two functions must also satisfy

$$\begin{bmatrix} \frac{\partial \delta_u}{\partial x_f} & \frac{\partial \delta_u}{\partial y_f} & \frac{\partial \delta_u}{\partial z_f} \end{bmatrix} j_{\Phi_f} = 0 \quad \text{(B-23)}$$

and

$$\begin{bmatrix} \frac{\partial z_p}{\partial x_f} & \frac{\partial z_p}{\partial y_f} & \frac{\partial z_p}{\partial z_f} \end{bmatrix} j_{\Phi_f} = 0 . \quad \text{(B-24)}$$

Expressing the perturbation in terms of the generalised coordinates used in the constant-velocity axes development (equation (7)), equation (B-17) becomes

$$\begin{bmatrix} e_u^{(c)} \\ f_u^{(c)} \\ g_u^{(c)} \end{bmatrix} = g_{u_f}^{(0)} \ell_{\Phi_f} - \left\{ \delta_u \ell_{\Phi_f} \ell_{\Phi_f}^T - g_{u_f}^{(0)} \ell_{\Phi_f} \left[\frac{\partial \delta_u}{\partial x_f}, \frac{\partial \delta_u}{\partial y_f}, \frac{\partial \delta_u}{\partial z_f} \right] \right.$$

$$\left. - \delta_u \ell_{\Phi_f} \left[\frac{\partial z_p}{\partial x_f}, \frac{\partial z_p}{\partial y_f}, \frac{\partial z_p}{\partial z_f} \right] \right\} [R \quad I \quad -A_{x_f}] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix}$$

$$= \text{(say)} \begin{bmatrix} e_{uf} \\ f_{uf} \\ g_{uf} \end{bmatrix} + \sum_{j=1}^{n+6} \begin{bmatrix} e_{uj} \\ f_{uj} \\ g_{uj} \end{bmatrix} q_j \quad . \tag{B-25}$$

The resultant overall forces and generalised forces can be obtained, as in the other cases, from the general expressions (74), (76), (22) and (78). We will write a generalised upholding (support) force as

$$- \left\{ (S_i)_f + \sum_{j=1}^{n+6} S_{ij} q_j \right\} \tag{B-26}$$

and the perturbation equation of motion will then be

$$\left\{ [A_{ij}] D^2 + [S_{ij}] + [G_{ij}] + [P_{ij}] + [E_{ij}] - [Q_{ij}] \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} = 0 \tag{B-27}$$

where the S_{ij} are obtained as above, the aerodynamic matrix $[Q_{ij}]$ is given in Table 15 and the other matrices are given in Table 3. The equation of equilibrium (Table 1) can similarly be modified.

Appendix C

PERTURBATION EQUATIONS OF MOTION IN THE GROUND RESONANCE TEST SITUATION

During a ground resonance test it is reasonable to assume that there will be no aerodynamic or propulsive forces, and the constant-velocity axes are stationary and such that $0_f y_f$ is coincident with the normal earth-fixed axis $0_0 y_0$. Thus the two attitude angles (Φ, Ψ) of the undisturbed aircraft are zero. There will of course normally be some excitation forces but as throughout the rest of this Report we will assume them to be zero. It will be a simple matter for the reader to include them if desired. We will also assume that at each support point the ground profile and stiffness are both stationary. Hence from (B-25) the upholding (support) force vector* is, referred to the constant-velocity axes

$$\begin{bmatrix} e_u^{(c)} \\ f_u^{(c)} \\ g_u^{(c)} \end{bmatrix} = g_{u_f}^{(0)} \ell_{\Phi_f} - \delta_u \ell_{\Phi_f} \ell_{\Phi_f}^T [R \quad I \quad -A_{x_f}] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \tag{C-1}$$

where, in this ground resonance test situation

$$\ell_{\Phi_f} = \begin{bmatrix} -\sin\Theta_f \\ 0 \\ \cos\Theta_f \end{bmatrix} . \tag{C-2}$$

From the equilibrium equation (*cf* Table 1) we immediately find that

$$-\sum g_{u_f}^{(0)} R^T \ell_{\Phi_f} = g \sum \delta m R^T \ell_{\Phi_f} + \sum R^T \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} \tag{C-3}$$

$$-\sum g_{u_f}^{(0)} = mg \tag{C-4}$$

and

$$\sum g_{u_f}^{(0)} A_{x_f} \ell_{\Phi_f} = 0 \tag{C-5}$$

* We are continuing to assume that the support forces are purely elastic. For the soft suspension desirable for a ground resonance test it may be necessary to take account also of the effect of the mass of the suspension on these support forces.

and so, using equations (75), (77) and (78) along with (C-1), the upholding (support) matrix is given by

$$-[S_{ij}] = -\sum\delta_u \begin{bmatrix} R^T \\ I \\ A_{x_f} \end{bmatrix} \ell_{\Phi_f} \ell^T_{\Phi_f} [R \quad I \quad -A_{x_f}]$$

$$+ \begin{bmatrix} 0 & 0 & -g\left(\sum\delta m R^T\right)A_{\ell_{\Phi_f}} - \sum R^T A_{e_{sf}} \\ 0 & 0 & 0 \\ gA_{\ell_{\Phi_f}} \sum\delta m R + \sum A_{e_{sf}} R & 0 & 0 \end{bmatrix} \quad \text{(C-6)}$$

where we have made use of the argument:-(C-3) is true for any R and in particular for $R^T = R^T A_\alpha$ where A_α is an arbitrary constant skew-symmetric matrix of the type of equation (6), and so

$$\left(-\sum(g^{(0)}_{u_f} + g\delta m)R^T\right)A_{\ell_{\Phi_f}} \begin{bmatrix} \alpha \\ \vdots \end{bmatrix} = \left(\sum R^T A_{e_{sf}}\right)\begin{bmatrix} \alpha \\ \vdots \end{bmatrix} \quad \text{(C-7)}$$

which gives since $\{\alpha \ldots\}$ is arbitrary

$$-\left(\sum g^{(0)}_{u_f} R^T\right)A_{\ell_{\Phi f}} = \sum R^T A_{e_{sf}} + g\left(\sum\delta m R^T\right)A_{\ell_{\Phi_f}} \quad . \quad \text{(C-8)}$$

The gravitational and structural matrices are given in Table 3, and so

$$[S_{ij}] + [G_{ij}] + [E_{ij}] = \sum\delta_u \begin{bmatrix} R^T \\ I \\ A_{x_f} \end{bmatrix} \ell_{\Phi_f} \ell^T_{\Phi_f} [R \quad I \quad -A_{x_f}]$$

$$- \begin{bmatrix} \sum R^T \begin{bmatrix} e_{s1} & \ldots & e_{sn} \\ f_{s1} & \ldots\ldots & \\ g_{s1} & \ldots\ldots & \end{bmatrix} & 0 & -\sum R^T A_{e_{sf}} \\ 0 & 0 & 0 \\ \sum A_{e_{sf}} R & 0 & 0 \end{bmatrix}$$

...... (C-9)

The equation of motion for small perturbations is therefore

$$\left\{[A_{ij}]D^2 + [S_{ij}] + [G_{ij}] + [E_{ij}]\right\} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} = 0 \qquad \text{(C-10)}$$

where the inertia matrix is given in Table 3 and the other terms come from the above equation (C-9). It is normal practice in ground resonance tests to make the support stiffnesses (the δ_u) as small as possible so that the first term in (C-9) can generally be ignored. We can also replace the terms involving the datum values of the structural force vectors by terms involving the coefficients in the expressions for the perturbations thereto. This is done through the relationship

$$\sum Ae_{sf}R = \sum Ax_f \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} \qquad \text{(C-11)}$$

which is a consequence of the fact that the structure cannot exert any overall moment on itself (*cf* the particular demonstration in Appendix A). It also should be noted that for most structures (*cf* Appendix A) the submatrix in (C-9)

$$\sum R^T \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix}$$

can be taken to be symmetric.

The main objective of a ground resonance test frequently is to gather information about the structural forces. If the function R represents the normal modes measured in such a test which are not predominantly body freedom modes, and the suspension is sufficiently soft then we see that to a good approximation

$$\sum \delta m R^T R \cdot \quad \text{is diagonal} \qquad \text{(C-12)}$$

$$\sum \delta m R = 0 \qquad \text{(C-13)}$$

$$\sum\delta mA_{x_f}R = \left(\sum A_{x_f}\begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix}\right)\left(\sum R^T\begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix}\right)^{-1}\left(\sum\delta mR^TR\right)$$

...... (C-14)

and

$$\sum R^T\begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} \text{ is diagonal.} \qquad \text{(C-15)}$$

Thus knowing the frequencies of these normal modes we can immediately find all the elements in the second term in (C-9), or in the structural matrix of Table 3, for these particular degrees of freedom and this particular datum motion. That is we have

$$\sum R^T\begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} \approx \left(\sum\delta mR^TR\right)\mathrm{diag}\{\omega_1^2 \ldots \omega_n^2\} \qquad \text{(C-16)}$$

$$\sum A_{x_f}\begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} \approx \left(\sum\delta mA_{x_f}R\right)\mathrm{diag}\{\omega_1^2 \ldots \omega_n^2\} \qquad \text{(C-17)}$$

where ω_i is the circular natural frequency of the ith mode. We would emphasise that these values of the structural coefficients strictly apply only to the ground resonance test situation. For other stress states during the datum motion it is obvious from (C-11) or (A-13) that $\begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix}$ will be different even if the modal matrix R is the same. Nevertheless, being devoid of hope of doing anything better without excessive effort, it will often be assumed that these coefficients, e_{si} etc, in the components of the local structural forces, are independent of the chosen unperturbed condition.

To obtain the structural matrix in the perturbation equations of motion using body-fixed axes is a little more difficult (*cf* Tables 4, 5, 6 and 7). We have from (131) and (46):

$$\sum R^T \begin{bmatrix} \hat{e}_{s1} & \cdots & \hat{e}_{sn} \\ \hat{f}_{s1} & \cdots\cdots & \\ \hat{g}_{s1} & \cdots\cdots & \end{bmatrix} = \sum R^T \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} + \left(\sum R^T A_{e_{sf}}\right) P_q$$

which, for normal modes, can be evaluated from (C-11), (C-16) and (C-17). The other terms (*cf*, for example, Table 4) comprise one, $P_q^T \sum A_{e_{sf}} R$, which similarly can be easily evaluated, and one $P_q^T\left(\sum A_{e_{sf}} A_{x_f}\right)P_q$ which sets a problem. To be able to evaluate the latter it appears that one will also require measurements of the unperturbed values of the upholding (support) forces. When using constant-velocity axes it was remarked above that one may often have to assume that $\begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix}$, and hence (*cf*, for example, (C-11)) the weighted sum $\sum A_{e_{sf}} R$ are independent of the chosen unperturbed condition. Similarly with body-fixed axes one may have to assume that $\begin{bmatrix} \hat{e}_{s1} & \cdots & \hat{e}_{sn} \\ \hat{f}_{s1} & \cdots\cdots & \\ \hat{g}_{s1} & \cdots\cdots & \end{bmatrix}$ and $\sum A_{e_{sf}} R + \left(\sum A_{x_f} A_{e_{sf}}\right) P_q$ are independent of the unperturbed state.

In section 7.4 we considered also the use of mean-body axes. It is however rather more difficult to relate the structural coefficients there derived (*cf* Table 14) to the results of a ground resonance test, for the normal modes obtained in a ground resonance test do not satisfy the two conditions (equation (177)) for mean-body axes (*cf* equations (C-13) and (C-14)). We will not therefore pursue the connection.

This consideration does however highlight the fact there is in general some inertia coupling between the normal modes measured in a ground resonance test and the rigid body rotational freedoms. How large this is it is difficult to tell. No doubt, for an almost two-dimensional structure, such as an aircraft, it could certainly be made small by the use of a continuous* soft support - from (C-11) and (C-14) it clearly will be small if the structural forces in the unperturbed state are small. In practice the cross inertia couplings between the measured modes and the rotational body freedoms have rarely been calculated. Moreover, if they had been, one would not know how much was the 'true' value, how much was due to experimental error (it is not unusual to find large inertia couplings between two measured 'normal' modes), and how much was due to

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* Continuous in the sense that every point on the underside of the aircraft is supported.

inadequacies in our mathematical model of the suspension (massless, elastic, and infinitely soft). All this, along with considerations of structural damping and friction, suggests that measured normal modes should be used with caution, and that one should regard ground resonance tests primarily as a fair check of theoretical predictions rather than a source of data.

Table 1

DATUM MOTION EQUATION USING CONSTANT-VELOCITY AXES

Equation

$$\left[-(Q_i)_f + (G_i)_f + (P_i)_f + (E_i)_f\right] = 0$$

Constituent matrices

Aerodynamic	Gravitational	Propulsive	Structural
$[(Q_i)_f]$	$-[(G_i)_f]$	$-[(P_i)_f]$	$-[(E_i)_f]$
$\begin{bmatrix} \sum R^T \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} \\ \begin{bmatrix} X_f \\ Y_f \\ Z_f \end{bmatrix} \\ \begin{bmatrix} L_f \\ M_f \\ N_f \end{bmatrix} \end{bmatrix}$	$\begin{bmatrix} g\left(\sum \delta m R^T\right) {}^{\ell}\Phi_f \\ mg\, {}^{\ell}\Phi_f \\ 0 \end{bmatrix}$	$\begin{bmatrix} \sum R^T \begin{bmatrix} e_{pf} \\ f_{pf} \\ g_{pf} \end{bmatrix} \\ \begin{bmatrix} X_{pf} \\ Y_{pf} \\ Z_{pf} \end{bmatrix} \\ \begin{bmatrix} L_{pf} \\ M_{pf} \\ N_{pf} \end{bmatrix} \end{bmatrix}$	$\begin{bmatrix} \sum R^T \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} \\ 0 \\ 0 \end{bmatrix}$

Overall forces

$$\begin{bmatrix} X_f \\ Y_f \\ Z_f \end{bmatrix} = \sum \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} \qquad \begin{bmatrix} L_f \\ M_f \\ N_f \end{bmatrix} = \sum A_{x_f} \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} \qquad \text{etc.}$$

Table 2

DATUM MOTION EQUATION USING BODY-FIXED AXES

Equation

$$\left[- (\hat{Q}_i)_f + (\hat{G}_i)_f + (\hat{P}_i)_f + (\hat{E}_i)_f \right] = 0$$

Constituent matrices

Aerodynamic	Gravitational	Propulsive	Structural
$[(\hat{Q}_i)_f]$	$-\ [(\hat{G}_i)_f]$	$-\ [(\hat{P}_i)_f]$	$-\ [(\hat{E}_i)_f]$
$\begin{bmatrix} \sum (R^T - R_0^T - P_q^T A x_f) \begin{bmatrix} e_f \\ f_f \\ g_f \end{bmatrix} \\ \begin{bmatrix} X_f \\ Y_f \\ Z_f \end{bmatrix} \\ \begin{bmatrix} L_f \\ M_f \\ N_f \end{bmatrix} \end{bmatrix}$	$\begin{bmatrix} g \left\{ \left(\sum \delta m R^T \right) - m R_0^T \right\} \ell_{\Phi_f} \\ mg \ell_{\Phi_f} \\ 0 \end{bmatrix}$	$\begin{bmatrix} \sum (R^T - R_0^T - P_q^T A x_f) \begin{bmatrix} e_{pf} \\ f_{pf} \\ g_{pf} \end{bmatrix} \\ \begin{bmatrix} X_{pf} \\ Y_{pf} \\ Z_{pf} \end{bmatrix} \\ \begin{bmatrix} L_{pf} \\ M_{pf} \\ N_{pf} \end{bmatrix} \end{bmatrix}$	$\begin{bmatrix} \sum R^T \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} \\ 0 \\ 0 \end{bmatrix}$

Overall forces

See Table 1

Note: For the purpose of this table equations (29), (30) and (31) have been combined into one matrix equation.

Table 3

PERTURBATION MOTION EQUATION USING CONSTANT-VELOCITY AXES

Equation

$$\left\{[A_{ij}]D^2 + [G_{ij}] + [P_{ij}] + [E_{ij}] - [Q_{ij}]\right\}\begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} = 0$$

Constituent matrices

(i) Inertia (ie ponderous inertia)

$$[A_{ij}] = \begin{bmatrix} \sum \delta m R^T R & \sum \delta m R^T & -\sum \delta m R^T A_{x_f} \\ \sum \delta m R & mI & 0 \\ \sum \delta m A_{x_f} R & 0 & I_n \end{bmatrix}$$

(ii) Gravitational

$$[G_{ij}] = -g \begin{bmatrix} 0 & 0 & \left(\sum \delta m R^T\right) A_{\ell_{\Phi_f}} \\ 0 & 0 & 0 \\ -A_{\ell_{\Phi_f}} \left(\sum \delta m R\right) & 0 & 0 \end{bmatrix}$$

(iii) Propulsive

$$[P_{ij}] = \begin{bmatrix} \left(\sum R^T A_{e_{pf}}\right) P_q & 0 & 0 \\ A_{x_{pf}} P_q & 0 & A_{x_{pf}} \\ \sum (A_{x_f} A_{e_{pf}} P_q + A_{e_{pf}} R) & 0 & \begin{bmatrix} 0 & 0 & 0 \\ N_{pf} & 0 & 0 \\ -M_{pf} & L_{pf} & 0 \end{bmatrix} \end{bmatrix}$$

Table 3 (concluded)

Constituent matrices (concluded)

(iv) Structural

$$[E_{ij}] = - \begin{bmatrix} \sum R^T \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(v) Aerodynamic

$$-[Q_{ij}] = - \begin{bmatrix} \sum R^T \begin{bmatrix} e_1 & \cdots & e_n \\ f_1 & \cdots\cdots & \\ g_1 & \cdots\cdots & \end{bmatrix} & \sum R^T \begin{bmatrix} e_{n\dot{+}1} & \cdots & e_{n\dot{+}3} \\ f_{n\dot{+}1} & \cdots\cdots & \\ g_{n\dot{+}1} & \cdots\cdots & \end{bmatrix} D & \sum R^T \left\{ \begin{bmatrix} e_{n\dot{+}1} & \cdots & e_{n\dot{+}3} \\ f_{n\dot{+}1} & \cdots\cdots & \\ g_{n\dot{+}1} & \cdots\cdots & \end{bmatrix} A u_f + \begin{bmatrix} e_{n\dot{+}4} & \cdots & e_{n\dot{+}6} \\ f_{n\dot{+}4} & \cdots\cdots & \\ g_{n\dot{+}4} & \cdots\cdots & \end{bmatrix} D \right\} \\ \begin{bmatrix} X_1^{(c)} & \cdots & X_n^{(c)} \\ Y_1^{(c)} & \cdots\cdots & \\ Z_1^{(c)} & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} X_{\dot{x}}^{(c)} & X_{\dot{y}}^{(c)} & X_{\dot{z}}^{(c)} \\ Y_{\dot{x}}^{(c)} & \cdots\cdots & \\ Z_{\dot{x}}^{(c)} & \cdots\cdots & \end{bmatrix} D & -A X_f + \begin{bmatrix} X_{\dot{x}}^{(c)} & X_{\dot{y}}^{(c)} & X_{\dot{z}}^{(c)} \\ Y_{\dot{x}}^{(c)} & \cdots\cdots & \\ Z_{\dot{x}}^{(c)} & \cdots\cdots & \end{bmatrix} A u_f + \begin{bmatrix} X_{\dot{\phi}}^{(c)} & X_{\dot{\theta}}^{(c)} & X_{\dot{\psi}}^{(c)} \\ Y_{\dot{\phi}}^{(c)} & \cdots\cdots & \\ Z_{\dot{\phi}}^{(c)} & \cdots\cdots & \end{bmatrix} D \\ \begin{bmatrix} L_1^{(c)} & \cdots & L_n^{(c)} \\ M_1^{(c)} & \cdots\cdots & \\ N_1^{(c)} & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} L_{\dot{x}}^{(c)} & L_{\dot{y}}^{(c)} & L_{\dot{z}}^{(c)} \\ M_{\dot{x}}^{(c)} & \cdots\cdots & \\ N_{\dot{x}}^{(c)} & \cdots\cdots & \end{bmatrix} D & - \begin{bmatrix} 0 & 0 & 0 \\ N_f & 0 & 0 \\ -M_f & L_f & 0 \end{bmatrix} + \begin{bmatrix} L_{\dot{x}}^{(c)} & L_{\dot{y}}^{(c)} & L_{\dot{z}}^{(c)} \\ M_{\dot{x}}^{(c)} & \cdots\cdots & \\ N_{\dot{x}}^{(c)} & \cdots\cdots & \end{bmatrix} A u_f + \begin{bmatrix} L_{\dot{\phi}}^{(c)} & L_{\dot{\theta}}^{(c)} & L_{\dot{\psi}}^{(c)} \\ M_{\dot{\phi}}^{(c)} & \cdots\cdots & \\ N_{\dot{\phi}}^{(c)} & \cdots\cdots & \end{bmatrix} D \end{bmatrix}$$

Overall aerodynamic force coefficients

See Table 8

Table 4

PERTURBATION MOTION EQUATION USING BODY-FIXED AXES IN TERMS OF ENCASTRÉ MODES AND DISPLACEMENT BODY FREEDOMS

Equation

$$\left\{[\hat{A}_{ij}]D^2 + [\hat{G}_{ij}] + [\hat{P}_{ij}] + [\hat{E}_{ij}] - [\hat{Q}_{ij}]\right\} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} = 0$$

Constituent matrices

(i) Inertia (ie ponderous inertia)

$$[\hat{A}_{ij}] = \begin{bmatrix} I & -R_0^T & -P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \sum \delta m R^T R & \sum \delta m R^T & -\sum \delta m R^T A_{x_f} \\ \sum \delta m R & mI & 0 \\ \sum \delta m A_{x_f} R & 0 & I_n \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -R_0 & I & 0 \\ -P_q & 0 & I \end{bmatrix}$$

(ii) Gravitational

$$[\hat{G}_{ij}] = \begin{bmatrix} 0 & 0 & -g\left(\sum \delta m R^T\right) A_{\ell_{\Phi_f}} + R_0^T A_{X_{gf}} \\ 0 & 0 & -A_{X_{gf}} \\ g A_{\ell_{\Phi_f}} \sum \delta m R - A_{X_{gf}} R_0 & 0 & 0 \end{bmatrix}$$

(iii) Propulsive

$$[\hat{P}_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sum A_{e_{pf}} R - A_{X_{pf}} R_0 + \left(\sum A_{e_{pf}} A_{X_f}\right) P_q & 0 & 0 \end{bmatrix}$$

Table 4 (concluded)

Constituent matrices (concluded)

(iv) Structural

$$[\hat{E}_{ij}] = \begin{bmatrix} -\sum R^T \begin{bmatrix} \hat{e}_{s1} & \cdots & \hat{e}_{sn} \\ \hat{f}_{s1} & \cdots\cdots & \\ \hat{g}_{s1} & \cdots\cdots & \end{bmatrix} + P_q^T \sum Ae_{sf} R + P_q^T \left(\sum Ae_{sf} A_{x_f} \right) P_q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(v) Aerodynamic

$$-[\hat{Q}_{\dot{i}\dot{j}}] = -\begin{bmatrix} I & -R_0^T & -P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \sum R^T \begin{bmatrix} \hat{e}_1 & \cdots & \hat{e}_n \\ \hat{f}_1 & \cdots\cdots & \\ \hat{g}_1 & \cdots\cdots & \end{bmatrix} & \sum R^T \begin{bmatrix} \hat{e}_{\dot{n+1}} & \cdots & \hat{e}_{\dot{n+3}} \\ \hat{f}_{\dot{n+1}} & \cdots\cdots & \\ \hat{g}_{\dot{n+1}} & \cdots\cdots & \end{bmatrix} D & \sum R^T \left\{ \begin{bmatrix} \hat{e}_{\dot{n+1}} & \cdots & \hat{e}_{\dot{n+3}} \\ \hat{f}_{\dot{n+1}} & \cdots\cdots & \\ \hat{g}_{\dot{n+1}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} \hat{e}_{\dot{n+4}} & \cdots & \hat{e}_{\dot{n+6}} \\ \hat{f}_{\dot{n+4}} & \cdots\cdots & \\ \hat{g}_{\dot{n+4}} & \cdots\cdots & \end{bmatrix} D \right\} \\ \begin{bmatrix} \hat{X}_1 & \cdots & \hat{X}_n \\ \hat{Y}_1 & \cdots\cdots & \\ \hat{Z}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{X}_{\dot{x}} & \hat{X}_{\dot{y}} & \hat{X}_{\dot{z}} \\ \hat{Y}_{\dot{x}} & \cdots\cdots & \\ \hat{Z}_{\dot{x}} & \cdots\cdots & \end{bmatrix} D & \begin{bmatrix} \hat{X}_{\dot{x}} & \hat{X}_{\dot{y}} & \hat{X}_{\dot{z}} \\ \hat{Y}_{\dot{x}} & \cdots\cdots & \\ \hat{Z}_{\dot{z}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} \hat{X}_{\dot{\phi}} & \hat{X}_{\dot{\theta}} & \hat{X}_{\dot{\psi}} \\ \hat{Y}_{\dot{\phi}} & \cdots\cdots & \\ \hat{Z}_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D \\ \begin{bmatrix} \hat{L}_1 & \cdots & \hat{L}_n \\ \hat{M}_1 & \cdots\cdots & \\ \hat{N}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{L}_{\dot{x}} & \hat{L}_{\dot{y}} & \hat{L}_{\dot{z}} \\ \hat{M}_{\dot{x}} & \cdots\cdots & \\ \hat{N}_{\dot{x}} & \cdots\cdots & \end{bmatrix} D & \begin{bmatrix} \hat{L}_{\dot{x}} & \hat{L}_{\dot{y}} & \hat{L}_{\dot{z}} \\ \hat{M}_{\dot{x}} & \cdots\cdots & \\ \hat{N}_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} \hat{L}_{\dot{\phi}} & \hat{L}_{\dot{\theta}} & \hat{L}_{\dot{\psi}} \\ \hat{M}_{\dot{\phi}} & \cdots\cdots & \\ \hat{N}_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D \end{bmatrix}$$

$$- \begin{bmatrix} P_q^T \left(A_{x_f} R_0 - \sum Ae_f R - \left(\sum Ae_f A_{x_f} \right) P_q \right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Overall aerodynamic force coefficients

See Table 9

Table 5

PERTURBATION MOTION EQUATION USING BODY-FIXED AXES IN TERMS OF ENCASTRÉ MODES AND VELOCITY BODY FREEDOMS

Equation

$$\left\{[\hat{A}_{ij}]D^2 + [\hat{J}_{ij}]D + [\hat{V}_{ij}] + [\hat{G}_{ij}] + [\hat{P}_{ij}] + [\hat{E}_{ij}] - [\hat{Q}_{ij}] + [\hat{G}^*_{ij}]D^{-1}\right\} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} = 0$$

Constituent matrices

(i) Ponderous

$$[\hat{A}_{ij}] = \begin{bmatrix} I & -R_0^T & -P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \sum \delta m R^T R & \sum \delta m R^T & -\sum \delta m R^T A_{x_f} \\ \sum \delta m R & mI & 0 \\ \sum \delta m A_{x_f} R & 0 & I_n \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -R_0 & 0 & 0 \\ -P_q & 0 & 0 \end{bmatrix}$$

$$[\hat{J}_{ij}] = \begin{bmatrix} I & -R_0^T & -P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & \sum \delta m R^T & -\sum \delta m R^T A_{x_f} \\ 0 & mI & 0 \\ 0 & 0 & I_n \end{bmatrix}$$

$$[\hat{V}_{ij}] = \begin{bmatrix} I & -R_0^T & -P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & -\left(\sum \delta m R^T\right) A_{u_f} \\ 0 & 0 & -m A_{u_f} \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) Gravitational

$$[\hat{G}_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g A_{\ell_{\Phi_f}} \sum \delta m R - A_{X_{gf}} R_0 & 0 & 0 \end{bmatrix}$$

$$[\hat{G}^*_{ij}] = \begin{bmatrix} 0 & 0 & -g\left(\sum \delta m R^T\right) A_{\ell_{\Phi_f}} + R_0^T A_{X_{gf}} \\ 0 & 0 & -A_{X_{gf}} \\ 0 & 0 & 0 \end{bmatrix}$$

077

Table 5 (continued)

Constituent matrices (continued)

(iii) Propulsive

$$[\hat{P}_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sum A_{e_{pf}} R - A_{x_{pf}} R_0 + \left(\sum A_{e_{pf}} A_{x_f}\right) P_q & 0 & 0 \end{bmatrix}$$

(iv) Structural

$$[\hat{E}_{ij}] = \begin{bmatrix} -\sum R^T \begin{bmatrix} \hat{e}_{s1} & \cdots & \hat{e}_{sn} \\ \hat{f}_{s1} & \cdots\cdots\cdots & \\ \hat{g}_{s1} & \cdots\cdots\cdots & \end{bmatrix} + P_q^T \sum A_{e_{sf}} R + P_q^T \left(\sum A_{e_{sf}} A_{x_f}\right) P_q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(v) Aerodynamic

$$-[\hat{Q}_{ij}] = -\begin{bmatrix} I & -R_0^T & -P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \sum R^T \begin{bmatrix} \hat{e}_1 & \cdots & \hat{e}_n \\ \hat{f}_1 & \cdots\cdots & \\ \hat{g}_1 & \cdots\cdots & \end{bmatrix} & \sum R^T \begin{bmatrix} \hat{e}_{n+1} & \cdots & \hat{e}_{n+3} \\ \hat{f}_{n+1} & \cdots\cdots\cdots & \\ \hat{g}_{n+1} & \cdots\cdots\cdots & \end{bmatrix} & \sum R^T \begin{bmatrix} \hat{e}_{n+4} & \cdots & \hat{e}_{n+6} \\ \hat{f}_{n+4} & \cdots\cdots\cdots & \\ \hat{g}_{n+4} & \cdots\cdots\cdots & \end{bmatrix} \\ \begin{bmatrix} \hat{X}_1 & \cdots & \hat{X}_n \\ \hat{Y}_1 & \cdots\cdots & \\ \hat{Z}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{X}_u & \hat{X}_v & \hat{X}_w \\ \hat{Y}_u & \cdots\cdots & \\ \hat{Z}_u & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{X}_p & \hat{X}_q & \hat{X}_r \\ \hat{Y}_p & \cdots\cdots & \\ \hat{Z}_p & \cdots\cdots & \end{bmatrix} \\ \begin{bmatrix} \hat{L}_1 & \cdots & \hat{L}_n \\ \hat{M}_1 & \cdots\cdots & \\ \hat{N}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{L}_u & \hat{L}_v & \hat{L}_w \\ \hat{M}_u & \cdots\cdots & \\ \hat{N}_u & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{L}_p & \hat{L}_q & \hat{L}_r \\ \hat{M}_p & \cdots\cdots & \\ \hat{N}_p & \cdots\cdots & \end{bmatrix} \end{bmatrix}$$

$$- \begin{bmatrix} P_q^T \left(A_{x_f} R_0 - \sum A_{e_f} R - \left(\sum A_{e_f} A_{x_f}\right) P_q\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

077

<u>Constituent matrices</u> (concluded)

<u>Overall aerodynamic force coefficients</u>

See Tables 9 and 10

Table 6

PERTURBATION MOTION EQUATION USING BODY-FIXED AXES IN TERMS OF FREE-FREE MODES AND DISPLACEMENT BODY FREEDOMS

Equation

$$\left\{[\tilde{A}_{ij}]D^2 + [\tilde{G}_{ij}] + [\tilde{P}_{ij}] + [\tilde{E}_{ij}] - [\tilde{Q}_{ij}]\right\} \begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_{n+6} \end{bmatrix} = 0$$

Constituent matrices

(i) Ponderous inertia

$$[\tilde{A}_{ij}] = \begin{bmatrix} \sum \delta m R^T R & \sum \delta m R^T & -\sum \delta m R^T A x_f \\ \sum \delta m R & mI & 0 \\ \sum \delta m A x_f R & 0 & I_n \end{bmatrix}$$

(ii) Gravitational

$$[\tilde{G}_{ij}] = \begin{bmatrix} g\left\{P_q^T A\ell_{\Phi_f}\left(\sum \delta m R\right) - \left(\sum \delta m R^T\right) A\ell_{\Phi_f} P_q\right\} - P_q^T A X_{gf} R_0 & 0 & -g\left(\sum \delta m R^T\right) A\ell_{\Phi_f} \\ -A X_{gf} P_q & 0 & -A X_{gf} \\ g A\ell_{\Phi_f}\left(\sum \delta m R\right) - A X_{gf} R_0 & 0 & 0 \end{bmatrix}$$

(iii) Propulsive

$$[\tilde{P}_{ij}] = \begin{bmatrix} P_q^T\left\{\sum A e_{pf} R + \left(\sum A e_{pf} A x_f\right) P_q - A X_{pf} R_0\right\} & 0 & 0 \\ 0 & 0 & 0 \\ \sum A e_{pf} R + \left(\sum A e_{pf} A x_f\right) P_q - A X_{pf} R_0 & 0 & 0 \end{bmatrix}$$

Table 6 (concluded)

Constituent matrices (concluded)

(iv) Structural

$$
[\tilde{E}_{\dot{i}\dot{j}}] = \begin{bmatrix} -\sum R^T \begin{bmatrix} \hat{e}_{s1} & \cdots & \hat{e}_{sn} \\ \hat{f}_{s1} & \cdots\cdots & \\ \hat{g}_{s1} & \cdots\cdots & \end{bmatrix} + P_q^T \left\{ \sum A_{e_{sf}} R + \left(\sum A_{e_{sf}} A_{x_f} \right) P_q \right\} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

(v) Aerodynamic

$$
-[\tilde{Q}_{\dot{i}\dot{j}}] = -\begin{bmatrix}
\sum R^T \begin{bmatrix} \tilde{e}_1 & \cdots & \tilde{e}_n \\ \tilde{f}_1 & \cdots\cdots & \\ \tilde{g}_1 & \cdots\cdots & \end{bmatrix} & \sum R^T \begin{bmatrix} \hat{e}_{\dot{n+1}} & \cdots & \hat{e}_{\dot{n+3}} \\ \hat{f}_{\dot{n+1}} & \cdots\cdots & \\ \hat{g}_{\dot{n+1}} & \cdots\cdots & \end{bmatrix} D & \sum R^T \left\{ \begin{bmatrix} \hat{e}_{\dot{n+1}} & \cdots & \hat{e}_{\dot{n+3}} \\ \hat{f}_{\dot{n+1}} & \cdots\cdots & \\ \hat{g}_{\dot{n+1}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} \hat{e}_{\dot{n+4}} & \cdots & \hat{e}_{\dot{n+6}} \\ \hat{f}_{\dot{n+4}} & \cdots\cdots & \\ \hat{g}_{\dot{n+4}} & \cdots\cdots & \end{bmatrix} D \right\} \\
\begin{bmatrix} \tilde{X}_1 & \cdots & \tilde{X}_n \\ \tilde{Y}_1 & \cdots\cdots & \\ \tilde{Z}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{X}_{\dot{x}} & \hat{X}_{\dot{y}} & \hat{X}_{\dot{z}} \\ \hat{Y}_{\dot{x}} & \cdots\cdots & \\ \hat{Z}_{\dot{x}} & \cdots\cdots & \end{bmatrix} D & \begin{bmatrix} \hat{X}_{\dot{x}} & \hat{X}_{\dot{y}} & \hat{X}_{\dot{z}} \\ \hat{Y}_{\dot{x}} & \cdots\cdots & \\ \hat{Z}_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} \hat{X}_{\dot{\phi}} & \hat{X}_{\dot{\theta}} & \hat{X}_{\dot{\psi}} \\ \hat{Y}_{\dot{\phi}} & \cdots\cdots & \\ \hat{Z}_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D \\
\begin{bmatrix} \tilde{L}_1 & \cdots & \tilde{L}_n \\ \tilde{M}_1 & \cdots\cdots & \\ \tilde{N}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{L}_{\dot{x}} & \hat{L}_{\dot{y}} & \hat{L}_{\dot{z}} \\ \hat{M}_{\dot{x}} & \cdots\cdots & \\ \hat{N}_{\dot{x}} & \cdots\cdots & \end{bmatrix} D & \begin{bmatrix} \hat{L}_{\dot{x}} & \hat{L}_{\dot{y}} & \hat{L}_{\dot{z}} \\ \hat{M}_{\dot{x}} & \cdots\cdots & \\ \hat{N}_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} \hat{L}_{\dot{\phi}} & \hat{L}_{\dot{\theta}} & \hat{L}_{\dot{\psi}} \\ \hat{M}_{\dot{\phi}} & \cdots\cdots & \\ \hat{N}_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D
\end{bmatrix}
$$

$$
- \begin{bmatrix} P_q^T \left(A_{x_f} R_0 - \sum A_{e_f} R - \left(\sum A_{e_f} A_{x_f} \right) P_q \right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

Overall aerodynamic force coefficients

See Tables 9 and 11

Table 7

PERTURBATION MOTION EQUATION USING BODY-FIXED AXES IN TERMS OF FREE-FREE MODES AND VELOCITY BODY FREEDOMS

Equation

$$\left\{[\check{A}_{ij}]D^2 + [\check{J}_{ij}]D + [\check{V}_{ij}] + [\check{G}_{ij}] + [\check{P}_{ij}] + [\check{E}_{ij}] - [\check{Q}_{ij}] + [\check{G}^*_{ij}]D^{-1}\right\}\begin{bmatrix}\check{q}_1\\ \vdots \\ \check{q}_{n+6}\end{bmatrix} = 0$$

Constituent matrices

(i) **Ponderous**

$$[\check{A}_{ij}] = \begin{bmatrix} \sum\delta mR^TR & 0 & 0 \\ \sum\delta mR & 0 & 0 \\ \sum\delta mA_{x_f}R & 0 & 0 \end{bmatrix}$$

$$[\check{J}_{ij}] = \begin{bmatrix} 0 & \sum\delta mR^T & -\sum\delta mR^TA_{x_f} \\ 0 & mI & 0 \\ 0 & 0 & I_n \end{bmatrix}$$

$$[\check{V}_{ij}] = \begin{bmatrix} 0 & 0 & -\left(\sum\delta mR^T\right)A_{u_f} \\ 0 & 0 & -mA_{u_f} \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) **Gravitational**

$$[\check{G}_{ij}] = \begin{bmatrix} g\left\{P_q^TA_{\ell_{\Phi_f}}\left(\sum\delta mR\right) - \left(\sum\delta mR^T\right)A_{\ell_{\Phi_f}}P_q\right\} - P_q^TA_{X_{gf}}R_0 & 0 & 0 \\ -A_{X_{gf}}P_q & 0 & 0 \\ gA_{\ell_{\Phi_f}}\left(\sum\delta mR\right) - A_{X_{gf}}R_0 & 0 & 0 \end{bmatrix}$$

Table 7 (continued)

Constituent matrices (continued)

(ii) Gravitational (concluded)

$$[\check{G}^*_{ij}] = \begin{bmatrix} 0 & 0 & -g\left(\sum \delta m R^T\right) A\ell_{\Phi_f} \\ 0 & 0 & -Ax_{gf} \\ 0 & 0 & 0 \end{bmatrix}$$

(iii) Propulsive

$$[\check{P}_{ij}] = \begin{bmatrix} P_q^T\left\{\sum Ae_{pf}R + \left(\sum Ae_{pf}Ax_f\right)P_q - Ax_{pf}R_0\right\} & 0 & 0 \\ 0 & 0 & 0 \\ \sum Ae_{pf}R + \left(\sum Ae_{pf}Ax_f\right)P_q - Ax_{pf}R_0 & 0 & 0 \end{bmatrix}$$

(iv) Structural

$$[\check{E}_{ij}] = \begin{bmatrix} -\sum R^T \begin{bmatrix} \hat{e}_{s1} & \cdots & \hat{e}_{sn} \\ \hat{f}_{s1} & \cdots\cdots & \\ \hat{g}_{s1} & \cdots\cdots & \end{bmatrix} + P_q^T\left\{\sum A_{e_{sf}}R + \left(\sum Ae_{sf}Ax_f\right)P_q\right\} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Table 7 (concluded)

Constituent matrices (concluded)

(v) Aerodynamics

$$
-\left[\breve{Q}_{ij}\right] = -\left[\begin{array}{ccc}
\sum R^T\begin{bmatrix}\breve{e}_1 & \cdots & \breve{e}_n\\ \breve{f}_1 & \cdots\cdots & \\ \breve{g}_1 & \cdots\cdots & \end{bmatrix} & \sum R^T\begin{bmatrix}\breve{e}_{n+1} & \cdots & \breve{e}_{n+3}\\ \breve{f}_{n+1} & \cdots\cdots & \\ \breve{g}_{n+1} & \cdots\cdots & \end{bmatrix} & \sum R^T\begin{bmatrix}\breve{e}_{n+4} & \cdots & \breve{e}_{n+6}\\ \breve{f}_{n+4} & \cdots\cdots & \\ \breve{g}_{n+4} & \cdots\cdots & \end{bmatrix}\\
\begin{bmatrix}\breve{X}_1 & \cdots & \breve{X}_n\\ \breve{Y}_1 & \cdots\cdots & \\ \breve{Z}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix}\hat{X}_u & \hat{X}_v & \hat{X}_w\\ \hat{Y}_u & \cdots\cdots & \\ \hat{Z}_u & \cdots\cdots & \end{bmatrix} & \begin{bmatrix}\hat{X}_p & \hat{X}_q & \hat{X}_r\\ \hat{Y}_p & \cdots\cdots & \\ \hat{Z}_p & \cdots\cdots & \end{bmatrix}\\
\begin{bmatrix}\breve{L}_1 & \cdots & \breve{L}_n\\ \breve{M}_1 & \cdots\cdots & \\ \breve{N}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix}\hat{L}_u & \hat{L}_v & \hat{L}_w\\ \hat{M}_u & \cdots\cdots & \\ \hat{N}_u & \cdots\cdots & \end{bmatrix} & \begin{bmatrix}\hat{L}_p & \hat{L}_q & \hat{L}_r\\ \hat{M}_p & \cdots\cdots & \\ \hat{N}_p & \cdots\cdots & \end{bmatrix}
\end{array}\right]
$$

$$
-\begin{bmatrix}
P_q^T\left(A_{X_f}R_0 - \sum A_{e_f}R - \left(\sum A_{e_f}A_{x_f}\right)P_q\right) & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0
\end{bmatrix}
$$

Overall aerodynamic force coefficients

See Tables 10 and 12

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Table 8

OVERALL AERODYNAMIC FORCE COEFFICIENTS - CONSTANT-VELOCITY AXES - WELL AWAY FROM GROUND

$$\begin{bmatrix} X_1^{(c)} & X_2^{(c)} & \ldots & X_n^{(c)} \\ Y_1^{(c)} & \ldots & \ldots & \ldots \\ Z_1^{(c)} & \ldots & \ldots & \ldots \end{bmatrix} = \sum \begin{bmatrix} e_1 & \ldots & e_n \\ f_1 & \ldots & \ldots \\ g_1 & \ldots & \ldots \end{bmatrix}$$

$$\begin{bmatrix} L_1^{(c)} & \ldots & L_n^{(c)} \\ M_1^{(c)} & \ldots & \ldots \\ N_1^{(c)} & \ldots & \ldots \end{bmatrix} = \sum A_{X_f} \begin{bmatrix} e_1 & \ldots & e_n \\ f_1 & \ldots & \ldots \\ g_1 & \ldots & \ldots \end{bmatrix} - \sum A_{ef} R$$

$$\begin{bmatrix} X_x^{(c)} & X_y^{(c)} & X_z^{(c)} \\ Y_x^{(c)} & \ldots & \ldots \\ Z_x^{(c)} & \ldots & \ldots \end{bmatrix} = \begin{bmatrix} X_{\dot{x}}^{(c)} & X_{\dot{y}}^{(c)} & X_{\dot{z}}^{(c)} \\ Y_{\dot{x}}^{(c)} & \ldots & \ldots \\ Z_{\dot{x}}^{(c)} & \ldots & \ldots \end{bmatrix} D$$

$$\begin{bmatrix} X_\phi^{(c)} & X_\theta^{(c)} & X_\psi^{(c)} \\ Y_\phi^{(c)} & \ldots & \ldots \\ Z_\phi^{(c)} & \ldots & \ldots \end{bmatrix} = - A_{X_f} + \begin{bmatrix} X_{\dot{x}}^{(c)} & X_{\dot{y}}^{(c)} & X_{\dot{z}}^{(c)} \\ Y_{\dot{x}}^{(c)} & \ldots & \ldots \\ Z_{\dot{x}}^{(c)} & \ldots & \ldots \end{bmatrix} A_{u_f} + \begin{bmatrix} X_{\dot{\phi}}^{(c)} & X_{\dot{\theta}}^{(c)} & X_{\dot{\psi}}^{(c)} \\ Y_{\dot{\phi}}^{(c)} & \ldots & \ldots \\ Z_{\dot{\phi}}^{(c)} & \ldots & \ldots \end{bmatrix} D$$

$$\begin{bmatrix} L_x^{(c)} & L_y^{(c)} & L_z^{(c)} \\ M_x^{(c)} & \ldots & \ldots \\ N_x^{(c)} & \ldots & \ldots \end{bmatrix} = - A_{X_f} + \begin{bmatrix} L_{\dot{x}}^{(c)} & L_{\dot{y}}^{(c)} & L_{\dot{z}}^{(c)} \\ M_{\dot{x}}^{(c)} & \ldots & \ldots \\ N_{\dot{x}}^{(c)} & \ldots & \ldots \end{bmatrix} D$$

$$\begin{bmatrix} L_\phi^{(c)} & L_\theta^{(c)} & L_\psi^{(c)} \\ M_\phi^{(c)} & \ldots & \ldots \\ N_\phi^{(c)} & \ldots & \ldots \end{bmatrix} = - A_{L_f} + \begin{bmatrix} L_{\dot{x}}^{(c)} & L_{\dot{y}}^{(c)} & L_{\dot{z}}^{(c)} \\ M_{\dot{x}}^{(c)} & \ldots & \ldots \\ N_{\dot{x}}^{(c)} & \ldots & \ldots \end{bmatrix} A_{u_f} + \begin{bmatrix} L_{\dot{\phi}}^{(c)} & L_{\dot{\theta}}^{(c)} & L_{\dot{\psi}}^{(c)} \\ M_{\dot{\phi}}^{(c)} & \ldots & \ldots \\ N_{\dot{\phi}}^{(c)} & \ldots & \ldots \end{bmatrix} D$$

Table 8 (concluded)

where $X_{\dot{x}}^{(c)} = \sum e_{\dot{n+1}}$ $\qquad X_{\dot{\phi}}^{(c)} = \sum e_{\dot{n+4}}$

$L_{\dot{x}}^{(c)} = \sum A_{x_f} e_{\dot{n+1}}$ $\qquad L_{\dot{\phi}}^{(c)} = \sum A_{x_f} e_{\dot{n+4}}$ etc.

NB: $X_{\dot{\phi}}^{(c)}$, for example, is not a derivative in the usual sense[2]. All the above terms such as $X_{\dot{y}}^{(c)}$, $X_{\dot{\phi}}^{(c)}$ are in general differential operators, and thus there will be contributions to the $X_{\dot{\phi}}$ derivative from $X_{\dot{y}}^{(c)}$, $X_{\dot{z}}^{(c)}$ and $X_{\dot{\phi}}^{(c)}$.

Table 9

OVERALL FORCE COEFFICIENTS - BODY-FIXED AXES - ENCASTRÉ MODES AND DISPLACEMENT BODY FREEDOMS

Gravitational

$$\begin{bmatrix} \hat{X}_{g1} & \cdots & \hat{X}_{g,n+6} \\ \hat{Y}_{g1} & \cdots & \cdots \\ \hat{Z}_{g1} & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} 0 & 0 & A_{X_{gf}} \end{bmatrix}$$

$$\begin{bmatrix} \hat{L}_{g1} & \cdots & \hat{L}_{g,n+6} \\ \hat{M}_{g1} & \cdots & \cdots \\ \hat{N}_{g1} & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} A_{X_{gf}} R_0 - g A_{\ell_{\Phi_f}} \sum \delta m R & 0 & 0 \end{bmatrix}$$

Propulsive

$$\begin{bmatrix} \hat{X}_{p1} & \cdots & \hat{X}_{p,n+6} \\ \hat{Y}_{p1} & \cdots & \cdots \\ \hat{Z}_{p1} & \cdots & \cdots \end{bmatrix} = 0$$

$$\begin{bmatrix} \hat{L}_{p1} & \cdots & \hat{L}_{p,n+6} \\ \hat{M}_{p1} & \cdots & \cdots \\ \hat{N}_{p1} & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} A_{X_{pf}} R_0 - \sum A_{e_{pf}} R + \left(\sum A_{e_{pf}} A_{x_f}\right) P_q & 0 & 0 \end{bmatrix}$$

Aerodynamic - well away from ground

$$\begin{bmatrix} \hat{X}_1 & \cdots & \hat{X}_n \\ \hat{Y}_1 & \cdots & \cdots \\ \hat{Z}_1 & \cdots & \cdots \end{bmatrix} = \sum \begin{bmatrix} \hat{e}_1 & \cdots & \hat{e}_n \\ \hat{f}_1 & \cdots & \cdots \\ \hat{g}_1 & \cdots & \cdots \end{bmatrix}$$

Table 9 (concluded)

Aerodynamic (concluded)

$$\begin{bmatrix} \hat{L}_1 & \cdots & \hat{L}_n \\ \hat{M}_1 & \cdots\cdots & \\ \hat{N}_1 & \cdots\cdots & \end{bmatrix} = \sum A_{x_f} \begin{bmatrix} \hat{e}_1 & \cdots & \hat{e}_n \\ \hat{f}_1 & \cdots\cdots & \\ \hat{g}_1 & \cdots\cdots & \end{bmatrix} - \sum A_{e_f} R + A_{x_f} R_0 - \left(\sum A_{e_f} A_{x_f}\right) P_q$$

$$\begin{bmatrix} \hat{X}_x & \hat{X}_y & \hat{X}_z \\ \hat{Y}_x & \cdots\cdots & \\ \hat{Z}_x & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{X}_{\dot{x}} & \hat{X}_{\dot{y}} & \hat{X}_{\dot{z}} \\ \hat{Y}_{\dot{x}} & \cdots\cdots & \\ \hat{Z}_{\dot{x}} & \cdots\cdots & \end{bmatrix} D$$

$$\begin{bmatrix} \hat{X}_\phi & \hat{X}_\theta & \hat{X}_\psi \\ \hat{Y}_\phi & \cdots\cdots & \\ \hat{Z}_\phi & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{X}_{\dot{x}} & \hat{X}_{\dot{y}} & \hat{X}_{\dot{z}} \\ \hat{Y}_{\dot{x}} & \cdots\cdots & \\ \hat{Z}_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} \hat{X}_{\dot{\phi}} & \hat{X}_{\dot{\theta}} & \hat{X}_{\dot{\psi}} \\ \hat{Y}_{\dot{\phi}} & \cdots\cdots & \\ \hat{Z}_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D$$

$$\begin{bmatrix} \hat{L}_x & \hat{L}_y & \hat{L}_z \\ \hat{M}_x & \cdots\cdots & \\ \hat{N}_x & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{L}_{\dot{x}} & \hat{L}_{\dot{y}} & \hat{L}_{\dot{z}} \\ \hat{M}_{\dot{x}} & \cdots\cdots & \\ \hat{N}_{\dot{x}} & \cdots\cdots & \end{bmatrix} D$$

$$\begin{bmatrix} \hat{L}_\phi & \hat{L}_\theta & \hat{L}_\psi \\ \hat{M}_\phi & \cdots\cdots & \\ \hat{N}_\phi & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{L}_{\dot{x}} & \hat{L}_{\dot{y}} & \hat{L}_{\dot{z}} \\ \hat{M}_{\dot{x}} & \cdots\cdots & \\ \hat{N}_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} \hat{L}_{\dot{\phi}} & \hat{L}_{\dot{\theta}} & \hat{L}_{\dot{\psi}} \\ \hat{M}_{\dot{\phi}} & \cdots\cdots & \\ \hat{N}_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D$$

where $\hat{X}_{\dot{x}} = \sum \hat{e}_{\dot{n+1}}$, $\quad \hat{X}_{\dot{\phi}} = \sum \hat{e}_{\dot{n+4}}$,

$\hat{L}_{\dot{x}} = \sum A_{x_f} \hat{e}_{\dot{n+1}}$, $\quad \hat{L}_{\dot{\phi}} = \sum A_{x_f} \hat{e}_{\dot{n+4}}$, etc.

NB: $\hat{X}_{\dot{\phi}}$, for example, is not the usual[2] $X_{\dot{\phi}}$ derivative. All the above terms such as $\hat{X}_{\dot{y}}$, $\hat{X}_{\dot{\phi}}$ are in general differential operators, and thus there will be contributions to the $X_{\dot{\phi}}$ derivative from $\hat{X}_{\dot{y}}$, $\hat{X}_{\dot{z}}$ and $\hat{X}_{\dot{\phi}}$.

Table 10

OVERALL AERODYNAMIC FORCE COEFFICIENTS FOR VELOCITY BODY FREEDOMS - BODY-FIXED AXES - WELL AWAY FROM GROUND

$$\begin{bmatrix} \hat{X}_u & \hat{X}_v & \hat{X}_w \\ \hat{Y}_u & \cdots\cdots & \\ \hat{Z}_u & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{X}_{\dot{x}} & \hat{X}_{\dot{y}} & \hat{X}_{\dot{z}} \\ \hat{Y}_{\dot{x}} & \cdots\cdots & \\ \hat{Z}_{\dot{x}} & \cdots\cdots & \end{bmatrix} = \sum \begin{bmatrix} \hat{e}_{n+1} & \cdots & \hat{e}_{n+3} \\ \hat{f}_{n+1} & \cdots\cdots & \\ \hat{g}_{n+1} & \cdots\cdots & \end{bmatrix}$$

$$\begin{bmatrix} \hat{X}_p & \hat{X}_q & \hat{X}_r \\ \hat{Y}_p & \cdots\cdots & \\ \hat{Z}_p & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{X}_{\dot{\phi}} & \hat{X}_{\dot{\theta}} & \hat{X}_{\dot{\psi}} \\ \hat{Y}_{\dot{\phi}} & \cdots\cdots & \\ \hat{Z}_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} = \sum \begin{bmatrix} \hat{e}_{n+4} & \cdots & \hat{e}_{n+6} \\ \hat{f}_{n+4} & \cdots\cdots & \\ \hat{g}_{n+4} & \cdots\cdots & \end{bmatrix}$$

$$\begin{bmatrix} \hat{L}_u & \hat{L}_v & \hat{L}_w \\ \hat{M}_u & \cdots\cdots & \\ \hat{N}_u & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{L}_{\dot{x}} & \hat{L}_{\dot{y}} & \hat{L}_{\dot{z}} \\ \hat{M}_{\dot{x}} & \cdots\cdots & \\ \hat{N}_{\dot{x}} & \cdots\cdots & \end{bmatrix} = \sum A x_f \begin{bmatrix} \hat{e}_{n+1} & \cdots & \hat{e}_{n+3} \\ \hat{f}_{n+1} & \cdots\cdots & \\ \hat{g}_{n+1} & \cdots\cdots & \end{bmatrix}$$

$$\begin{bmatrix} \hat{L}_p & \hat{L}_q & \hat{L}_r \\ \hat{M}_p & \cdots\cdots & \\ \hat{N}_p & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} \hat{L}_{\dot{\phi}} & \hat{L}_{\dot{\theta}} & \hat{L}_{\dot{\psi}} \\ \hat{M}_{\dot{\phi}} & \cdots\cdots & \\ \hat{N}_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} = \sum A x_f \begin{bmatrix} \hat{e}_{n+4} & \cdots & \hat{e}_{n+6} \\ \hat{f}_{n+4} & \cdots\cdots & \\ \hat{g}_{n+4} & \cdots\cdots & \end{bmatrix}$$

(*cf* equations (101) and (102), and note to Table 9)

Table 11

OVERALL FORCE COEFFICIENTS - BODY-FIXED AXES - FREE-FREE MODES AND DISPLACEMENT BODY FREEDOMS

Gravitational

$$\begin{bmatrix} \tilde{X}_{g1} & \cdots & \tilde{X}_{g,n+6} \\ \tilde{Y}_{g1} & \cdots & \cdots \\ \tilde{Z}_{g1} & \cdots & \cdots \end{bmatrix} = [A_{X_{gf}} P_q \quad 0 \quad A_{X_{gf}}]$$

$$\begin{bmatrix} \tilde{L}_{g1} & \cdots & \tilde{L}_{g,n+6} \\ \tilde{M}_{g1} & \cdots & \cdots \\ \tilde{N}_{g1} & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} A_{X_{gf}} R_0 - g A_{\ell_{\Phi_f}} \sum \delta m R & 0 & 0 \end{bmatrix}$$

Propulsive

$$\begin{bmatrix} \tilde{X}_{p1} & \cdots & \tilde{X}_{p,n+6} \\ \tilde{Y}_{p1} & \cdots & \cdots \\ \tilde{Z}_{p1} & \cdots & \cdots \end{bmatrix} = 0$$

$$\begin{bmatrix} \tilde{L}_{p1} & \cdots & \tilde{L}_{p,n+6} \\ \tilde{M}_{p1} & \cdots & \cdots \\ \tilde{N}_{p1} & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} A_{X_{pf}} R_0 - \sum A_{e_{pf}} R + \left(\sum A_{e_{pf}} A_{x_f}\right) P_q & 0 & 0 \end{bmatrix}$$

Aerodynamic - well away from ground

$$\begin{bmatrix} \tilde{X}_1 & \cdots & \tilde{X}_n \\ \tilde{Y}_1 & \cdots & \cdots \\ \tilde{Z}_1 & \cdots & \cdots \end{bmatrix} = \sum \begin{bmatrix} \tilde{e}_1 & \cdots & \tilde{e}_n \\ \tilde{f}_1 & \cdots & \cdots \\ \tilde{g}_1 & \cdots & \cdots \end{bmatrix}$$

Table 11 (concluded)

Aerodynamic (concluded)

$$\begin{bmatrix} \tilde{L}_1 & \cdots & \tilde{L}_n \\ \tilde{M}_1 & \cdots\cdots\cdots & \\ \tilde{N}_1 & \cdots\cdots\cdots & \end{bmatrix} = \sum A_{x_f} \begin{bmatrix} \tilde{e}_1 & \cdots & \tilde{e}_n \\ \tilde{f}_1 & \cdots\cdots\cdots & \\ \tilde{g}_1 & \cdots\cdots\cdots & \end{bmatrix} - \sum A_{e_f} R + A_{X_f} R_0 - \left(\sum A_{e_f} A_{x_f}\right) P_q \ .$$

For the other aerodynamic coefficients see Table 9 since $\tilde{X}_x = \hat{X}_x$, $\tilde{X}_\phi = \hat{X}_\phi$ etc.

Table 12

OVERALL AERODYNAMIC FORCE COEFFICIENTS - BODY-FIXED AXES - FREE-FREE MODES - VELOCITY BODY FREEDOMS - WELL AWAY FROM GROUND

$$\begin{bmatrix} \breve{X}_1 & \cdots & \breve{X}_n \\ \breve{Y}_1 & \cdots\cdots\cdots & \\ \breve{Z}_1 & \cdots\cdots\cdots & \end{bmatrix} = \sum \begin{bmatrix} \breve{e}_1 & \cdots & \breve{e}_n \\ \breve{f}_1 & \cdots\cdots\cdots & \\ \breve{g}_1 & \cdots\cdots\cdots & \end{bmatrix}$$

$$\begin{bmatrix} \breve{L}_1 & \cdots & \breve{L}_n \\ \breve{M}_1 & \cdots\cdots\cdots & \\ \breve{N}_1 & \cdots\cdots\cdots & \end{bmatrix} = \sum A_{x_f} \begin{bmatrix} \breve{e}_1 & \cdots & \breve{e}_n \\ \breve{f}_1 & \cdots\cdots\cdots & \\ \breve{g}_1 & \cdots\cdots\cdots & \end{bmatrix} - \sum A_{e_f} R + A_{x_f} R_0 - \left(\sum A_{e_f} A_{x_f}\right) P_q$$

For other aerodynamic coefficients see Table 10 since $\breve{X}_u = \hat{X}_u$, $\breve{X}_p = \hat{X}_p$ etc.

Table 13

TRANSFORMATION BETWEEN DIFFERENT FORMS OF THE PERTURBATION EQUATION OF MOTION

To transform from one form of the perturbation equation to another one requires, in addition to the coefficients in the original equation, the following quantities (*cf* section 4.3):

From (or to)	To (or from)	Form	Constant-velocity	Body-fixed			
				Encastré		Free-free	
				Disp.	Velocity	Disp.	Velocity
				Dressing			
Form		Dressing		^	⌒	~	⌣
Constant-velocity							
Body-fixed	Encastré displacement	^	R_0, P_q $\bar{x}_f$, $\bar{L}_f$ etc $\sum A\bar{z}_f R$, $\sum A\bar{z}_f A_{x_f}$			SEE OPPOSITE	
Body-fixed	Encastré velocity	⌒	R_0, P_q, A_{u_f} $\bar{x}_f$, $\bar{L}_f$, etc $\sum A\bar{z}_f R$, $\sum A\bar{z}_f A_{x_f}$	A_{u_f}			
Body-fixed	Free-free displacement	~	R_0, P_q $\bar{x}_f$, $\bar{L}_f$, etc $\sum A\bar{z}_f R$, $\sum A\bar{z}_f A_{x_f}$	R_0, P_q	R_0, P_q A_{u_f}		
Body-fixed	Free-free velocity	⌣	R_0, P_q, A_{u_f} $\bar{x}_f$, $\bar{L}_f$, etc $\sum A\bar{z}_f R$, $\sum A\bar{z}_f A_{x_f}$	R_0, P_q A_{u_f}	R_0, P_q	A_{u_f}	

Notes: (i) A_{u_f} should always be known.

(ii) Need information about deformation modes going from or to to deduce R_0, P_q.

(iii) Entries in the table of a typical force vector indicate the complete set of these (aerodynamic, propulsive etc) are required.

(iv) In transformation from or to the constant-velocity form the datum-motion forces could be considered zero. The result would be an equation with the wrong constituent coefficients but the right solution.

..ble 14

PERTURBATION MOTION EQUATION USING MEAN-BODY AXES IN TERMS OF FREE-FREE MODES AND VELOCITY BODY FREEDOMS

Equation

$$\left\{[\bar{\bar{A}}_{ij}]D^2 + [\bar{\bar{J}}_{ij}]D + [\bar{\bar{V}}_{ij}] + [\bar{\bar{P}}_{ij}] + [\bar{\bar{E}}_{ij}] - [\bar{\bar{Q}}_{ij}] + [\bar{\bar{G}}^*_{ij}]D^{-1}\right\}\begin{bmatrix} \check{q}_1 \\ \vdots \\ \check{q}_{n+6} \end{bmatrix} = 0$$

Constituent matrices

(i) **Ponderous**

$$[\bar{\bar{A}}_{ij}] = \mathrm{diag}\left\{\sum \delta m R^T R \quad 0 \quad 0\right\}$$

$$[\bar{\bar{J}}_{ij}] = \mathrm{diag}\left\{0 \quad mI \quad I_n\right\}$$

$$[\bar{\bar{V}}_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -mA_{uf} \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) **Gravitational**

$$[\bar{\bar{G}}^*_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -Ax_{gf} \\ 0 & 0 & 0 \end{bmatrix}$$

(iii) **Propulsive**

$$[\bar{\bar{P}}_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sum Ae_{pf}R + \left(\sum Ax_f Ae_{pf}\right)P_q & 0 & 0 \end{bmatrix}$$

Table 14 (concluded)

Constituent matrices (concluded)

(iv) Structural

$$[\bar{\bar{E}}_{ij}] = \begin{bmatrix} -\sum R^T \begin{bmatrix} \hat{e}_{s1} & \cdots & \hat{e}_{sn} \\ \hat{f}_{s1} & \cdots\cdots & \\ \hat{g}_{s1} & \cdots\cdots & \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(v) Aerodynamic

$$-[\bar{\bar{Q}}_{ij}] = -\begin{bmatrix} \sum R^T \begin{bmatrix} \check{e}_1 & \cdots & \check{e}_n \\ \check{f}_1 & \cdots\cdots & \\ \check{g}_1 & \cdots\cdots & \end{bmatrix} & \sum R^T \begin{bmatrix} \check{e}_{n+1} & \cdots & \check{e}_{n+3} \\ \check{f}_{n+1} & \cdots\cdots & \\ \check{g}_{n+1} & \cdots\cdots & \end{bmatrix} & \sum R^T \begin{bmatrix} \check{e}_{n+4} & \cdots & \check{e}_{n+6} \\ \check{f}_{n+4} & \cdots\cdots & \\ \check{g}_{n+4} & \cdots\cdots & \end{bmatrix} \\ \begin{bmatrix} X_1^{(c)} & \cdots & X_n^{(c)} \\ Y_1^{(c)} & \cdots\cdots & \\ Z_1^{(c)} & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{X}_u & \hat{X}_v & \hat{X}_w \\ \hat{Y}_u & \cdots\cdots & \\ \hat{Z}_u & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{X}_p & \hat{X}_q & \hat{X}_r \\ \hat{Y}_p & \cdots\cdots & \\ \hat{Z}_p & \cdots\cdots & \end{bmatrix} \\ \begin{bmatrix} L_1^{(c)} & \cdots & L_n^{(c)} \\ M_1^{(c)} & \cdots\cdots & \\ N_1^{(c)} & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{L}_u & \hat{L}_v & \hat{L}_w \\ \hat{M}_u & \cdots\cdots & \\ \hat{N}_u & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{L}_p & \hat{L}_q & \hat{L}_r \\ \hat{M}_p & \cdots\cdots & \\ \hat{N}_p & \cdots\cdots & \end{bmatrix} \end{bmatrix}$$

NB: In the above equations the modal matrix R has been chosen such that

$$\left\{ \begin{aligned} \sum \delta m R &= 0 \\ \sum \delta m A_{x_f} R &= 0 \end{aligned} \right.$$

These equations of motion have not been derived directly in the present paper but in section 7.4 we have merely shown that they are equivalent, for a modal matrix
077 satisfying the above conditions, to the equations of Table 7.

Table 15

AERODYNAMIC MATRICES IN PERTURBATION EQUATIONS OF MOTION WHEN NEAR THE GROUND

Constant-velocity axes

$$
-[Q_{ij}] = -\begin{bmatrix}
\sum R^T\begin{bmatrix} e_1 & \cdots & e_n \\ f_1 & \cdots & \\ g_1 & \cdots & \end{bmatrix} &
\sum R^T\begin{bmatrix} e_{n+1} & \cdots & e_{n+3} \\ f_{n+1} & \cdots & \\ g_{n+1} & \cdots & \end{bmatrix} &
\sum R^T\left\{\begin{bmatrix} e_{n+4} & \cdots & e_{n+6} \\ f_{n+4} & \cdots & \\ g_{n+4} & \cdots & \end{bmatrix} + A_{e_f}\right\} \\
\begin{bmatrix} X_1^{(c)} & \cdots & X_n^{(c)} \\ Y_1^{(c)} & \cdots & \\ Z_1^{(c)} & \cdots & \end{bmatrix} &
\begin{bmatrix} X_x^{(c)} & \cdots & X_z^{(c)} \\ Y_x^{(c)} & \cdots & \\ Z_x^{(c)} & \cdots & \end{bmatrix} &
\begin{bmatrix} X_\phi^{(c)} & X_\theta^{(c)} & X_\psi^{(c)} \\ Y_\phi^{(c)} & \cdots & \\ Z_\phi^{(c)} & \cdots & \end{bmatrix} \\
\begin{bmatrix} L_1^{(c)} & \cdots & L_n^{(c)} \\ M_1^{(c)} & \cdots & \\ N_1^{(c)} & \cdots & \end{bmatrix} &
\begin{bmatrix} L_x^{(c)} & \cdots & L_z^{(c)} \\ M_x^{(c)} & \cdots & \\ N_x^{(c)} & \cdots & \end{bmatrix} + A x_f &
\begin{bmatrix} L_\phi^{(c)} & L_\theta^{(c)} & L_\psi^{(c)} \\ M_\phi & \cdots & \\ N_\phi & \cdots & \end{bmatrix} + \begin{bmatrix} 0 & -N_f & M_f \\ 0 & 0 & -L_f \\ 0 & 0 & 0 \end{bmatrix}
\end{bmatrix}
$$

*Body-fixed axes, encastré modes, displacement body freedoms

$$
-[\hat{Q}_{ij}] = -\begin{bmatrix} I & -R_0^T & -P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}
\begin{bmatrix}
\sum R^T\begin{bmatrix} \hat{e}_1 & \cdots & \hat{e}_n \\ \hat{f}_1 & \cdots & \\ \hat{g}_1 & \cdots & \end{bmatrix} &
\sum R^T\begin{bmatrix} \hat{e}_{n+1} & \cdots & \hat{e}_{n+3} \\ \hat{f}_{n+1} & \cdots & \\ \hat{g}_{n+1} & \cdots & \end{bmatrix} &
\sum R^T\begin{bmatrix} \hat{e}_{n+4} & \cdots & \hat{e}_{n+6} \\ \hat{f}_{n+4} & \cdots & \\ \hat{g}_{n+4} & \cdots & \end{bmatrix} \\
\begin{bmatrix} \hat{X}_1 & \cdots & \hat{X}_n \\ \hat{Y}_1 & \cdots & \\ \hat{Z}_1 & \cdots & \end{bmatrix} &
\begin{bmatrix} \hat{X}_x & \hat{X}_y & \hat{X}_z \\ \hat{Y}_x & \cdots & \\ \hat{Z}_x & \cdots & \end{bmatrix} &
\begin{bmatrix} \hat{X}_\phi & \hat{X}_\theta & \hat{X}_\psi \\ \hat{Y}_\phi & \cdots & \\ \hat{Z}_\phi & \cdots & \end{bmatrix} \\
\begin{bmatrix} \hat{L}_1 & \cdots & \hat{L}_n \\ \hat{M}_1 & \cdots & \\ \hat{N}_1 & \cdots & \end{bmatrix} &
\begin{bmatrix} \hat{L}_x & \hat{L}_y & \hat{L}_z \\ \hat{M}_x & \cdots & \\ \hat{N}_x & \cdots & \end{bmatrix} &
\begin{bmatrix} \hat{L}_\phi & \hat{L}_\theta & \hat{L}_\psi \\ \hat{M}_\phi & \cdots & \\ \hat{N}_\phi & \cdots & \end{bmatrix}
\end{bmatrix} - [\&]
$$

Table 15 (continued)

*Body-fixed axes, encastré modes, velocity body freedoms

$$
-[\hat{Q}_{ij}] = -\begin{bmatrix} I & -R_0^T & -P_q^T \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}
\begin{bmatrix}
\sum R^T \begin{bmatrix} \hat{e}_1 & \cdots & \hat{e}_n \\ \hat{f}_1 & \cdots\cdots & \\ \hat{g}_1 & \cdots\cdots & \end{bmatrix} & \sum R^T \begin{bmatrix} \hat{e}_{n+1} & \cdots & \hat{e}_{n+3} \\ \hat{f}_{n+1} & \cdots\cdots & \\ \hat{g}_{n+1} & \cdots\cdots & \end{bmatrix} D^{-1} & \sum R^T \left\{ -\begin{bmatrix} \hat{e}_{n+1} & \cdots & \hat{e}_{n+3} \\ \hat{f}_{n+1} & \cdots\cdots & \\ \hat{g}_{n+1} & \cdots\cdots & \end{bmatrix} A_{uf} D^{-2} + \begin{bmatrix} \hat{e}_{n+4} & \cdots & \hat{e}_{n+6} \\ \hat{f}_{n+4} & \cdots\cdots & \\ \hat{g}_{n+4} & \cdots\cdots & \end{bmatrix} D^{-1} \right\} \\
\begin{bmatrix} \hat{X}_1 & \cdots & \hat{X}_n \\ \hat{Y}_1 & \cdots\cdots & \\ \hat{Z}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{X}_x & \hat{X}_y & \hat{X}_z \\ \hat{Y}_x & \cdots\cdots & \\ \hat{Z}_x & \cdots\cdots & \end{bmatrix} D^{-1} & -\begin{bmatrix} \hat{X}_x & \hat{X}_y & \hat{X}_z \\ \hat{Y}_x & \cdots\cdots & \\ \hat{Z}_x & \cdots\cdots & \end{bmatrix} A_{uf} D^{-2} + \begin{bmatrix} \hat{X}_\phi & \hat{X}_\theta & \hat{X}_\psi \\ \hat{Y}_\phi & \cdots\cdots & \\ \hat{Z}_\phi & \cdots\cdots & \end{bmatrix} D^{-1} \\
\begin{bmatrix} \hat{L}_1 & \cdots & \hat{L}_n \\ \hat{M}_1 & \cdots\cdots & \\ \hat{N}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{L}_x & \hat{L}_y & \hat{L}_z \\ \hat{M}_x & \cdots\cdots & \\ \hat{N}_x & \cdots\cdots & \end{bmatrix} D^{-1} & -\begin{bmatrix} \hat{L}_x & \hat{L}_y & \hat{L}_z \\ \hat{M}_x & \cdots\cdots & \\ \hat{N}_x & \cdots\cdots & \end{bmatrix} A_{uf} D^{-2} + \begin{bmatrix} \hat{L}_\phi & \hat{L}_\theta & \hat{L}_\psi \\ \hat{M}_\phi & \cdots\cdots & \\ \hat{N}_\phi & \cdots\cdots & \end{bmatrix} D^{-1}
\end{bmatrix} - [\&]
$$

*Body-fixed axes, free-free modes, displacement body freedoms

$$
-[\tilde{Q}_{ij}] = -\begin{bmatrix}
\sum R^T \begin{bmatrix} \tilde{e}_1 & \cdots & \tilde{e}_n \\ \tilde{f}_1 & \cdots\cdots & \\ \tilde{g}_1 & \cdots\cdots & \end{bmatrix} & \sum R^T \begin{bmatrix} \hat{e}_{n+1} & \cdots & \hat{e}_{n+3} \\ \hat{f}_{n+1} & \cdots\cdots & \\ \hat{g}_{n+1} & \cdots\cdots & \end{bmatrix} & \sum R^T \begin{bmatrix} \hat{e}_{n+4} & \cdots & \hat{e}_{n+6} \\ \hat{f}_{n+4} & \cdots\cdots & \\ \hat{g}_{n+4} & \cdots\cdots & \end{bmatrix} \\
\begin{bmatrix} \tilde{X}_1 & \cdots & \tilde{X}_n \\ \tilde{Y}_1 & \cdots\cdots & \\ \tilde{Z}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{X}_x & \hat{X}_y & \hat{X}_z \\ \hat{Y}_x & \cdots\cdots & \\ \hat{Z}_x & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{X}_\phi & \hat{X}_\theta & \hat{X}_\psi \\ \hat{Y}_\phi & \cdots\cdots & \\ \hat{Z}_\phi & \cdots\cdots & \end{bmatrix} \\
\begin{bmatrix} \tilde{L}_1 & \cdots & \tilde{L}_n \\ \tilde{M}_1 & \cdots\cdots & \\ \tilde{N}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{L}_x & \hat{L}_y & \hat{L}_z \\ \hat{M}_x & \cdots\cdots & \\ \hat{N}_x & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} \hat{L}_\phi & \hat{L}_\theta & \hat{L}_\psi \\ \hat{M}_\phi & \cdots\cdots & \\ \hat{N}_\phi & \cdots\cdots & \end{bmatrix}
\end{bmatrix} - [\&]
$$

Table 15 (concluded)

***Body-fixed axes, free-free modes, velocity body freedoms**

$$
-[\breve{Q}_{ij}] = -\begin{bmatrix}
\sum R^T\begin{bmatrix}\tilde{e}_1 & \cdots & \tilde{e}_n\\ \tilde{f}_1 & \cdots\cdots & \\ \tilde{g}_1 & \cdots\cdots & \end{bmatrix} & \sum R^T\begin{bmatrix}\hat{e}_{n+1} & \cdots & \hat{e}_{n+3}\\ \hat{f}_{n+1} & \cdots\cdots & \\ \hat{g}_{n+1} & \cdots\cdots & \end{bmatrix}D^{-1} & \sum R^T\left\{-\begin{bmatrix}\hat{e}_{n+1} & \cdots & \hat{e}_{n+3}\\ \hat{f}_{n+1} & \cdots\cdots & \\ \hat{g}_{n+1} & \cdots\cdots & \end{bmatrix}A_{u_f}D^{-2} + \begin{bmatrix}\hat{e}_{n+4} & \cdots & \hat{e}_{n+6}\\ \hat{f}_{n+4} & \cdots\cdots & \\ \hat{g}_{n+4} & \cdots\cdots & \end{bmatrix}D^{-1}\right\} \\
\begin{bmatrix}\tilde{X}_1 & \cdots & \tilde{X}_n\\ \tilde{Y}_1 & \cdots\cdots & \\ \tilde{Z}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix}\hat{X}_x & \hat{X}_y & \hat{X}_z\\ \hat{Y}_x & \cdots\cdots & \\ \hat{Z}_x & \cdots\cdots & \end{bmatrix}D^{-1} & -\begin{bmatrix}\hat{X}_x & \hat{X}_y & \hat{X}_z\\ \hat{Y}_x & \cdots\cdots & \\ \hat{Z}_x & \cdots\cdots & \end{bmatrix}A_{u_f}D^{-2} + \begin{bmatrix}\hat{X}_\phi & \hat{X}_\theta & \hat{X}_\psi\\ \hat{Y}_\phi & \cdots\cdots & \\ \hat{Z}_\phi & \cdots\cdots & \end{bmatrix}D^{-1} \\
\begin{bmatrix}\tilde{L}_1 & \cdots & \tilde{L}_n\\ \tilde{M}_1 & \cdots\cdots & \\ \tilde{N}_1 & \cdots\cdots & \end{bmatrix} & \begin{bmatrix}\hat{L}_x & \hat{L}_y & \hat{L}_z\\ \hat{M}_x & \cdots\cdots & \\ \hat{N}_x & \cdots\cdots & \end{bmatrix}D^{-1} & -\begin{bmatrix}\hat{L}_x & \hat{L}_y & \hat{L}_z\\ \hat{M}_x & \cdots\cdots & \\ \hat{N}_x & \cdots\cdots & \end{bmatrix}A_{u_f}D^{-2} + \begin{bmatrix}\hat{L}_\phi & \hat{L}_\theta & \hat{L}_\psi\\ \hat{M}_\phi & \cdots\cdots & \\ \hat{N}_\phi & \cdots\cdots & \end{bmatrix}D^{-1}
\end{bmatrix} - [\&]
$$

* In the above expressions:

$$
[\&] \equiv \begin{bmatrix}
P_q^T\left(A_{X_f}R_0 - \sum A_{e_f}R - \left(\sum A_{e_f}A_{x_f}\right)P_q\right) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

GLOSSARY OF TERMS

(i) Fundamental terms

degree of freedom	a system is said to have n degrees of freedom if its position and shape can be uniquely specified by n parameters, but not by less than n
semi-rigid	a body is said to be semi-rigid if it has a finite number of degrees of freedom in addition to any body freedoms
body freedom	a body is said to have n body freedoms ($n \leq 6$) if its position, with a given shape, can be uniquely specified by n parameters but not by less than n
perturbation	a disturbance from a datum state
deformation	a perturbation in shape
circular frequency	the frequency of oscillation multiplied by 2π
natural frequency	a constant frequency of free oscillation of a system
generalised coordinates	the set of parameters used to specify the position and shape of a system
generalised force	the coefficient of the increment of a generalised coordinate in the expression for the virtual work of a system

(ii) Types of force

aerodynamic*	exerted by the air on the external surface of a body
propulsive*	produced by the operation of a propulsive unit such as a jet engine
gravitational	resulting from the gravitational attraction between each particle and the earth
structural	resulting from the stresses in a body
upholding (or support)	resulting from contact between a body and the ground
ponderous	a reversed effective force consequent to a system having mass
inertia	a force which is proportional to the second derivative, with respect to time, of a generalised coordinate
damping	a force which is proportional to the first derivative, with respect to time, of a generalised coordinate

* The distinction between aerodynamic and propulsive is not entirely clear-cut.

GLOSSARY OF TERMS (continued)

(ii) (concluded)

stiffness	a force which is proportional to a generalised coordinate

(iii) Frames of reference (all right-handed orthogonal* cartesian)

normal earth-fixed axes	axes fixed relative to the earth with the z-axis vertically downwards
constant-velocity axes	axes having constant linear and angular velocity relative to an inertial frame. (The ones used in this Report have zero angular velocity.) (For the datum motion considered in this Report these are always taken to be coincident with the datum-attitude earth axes - see below.)
body-fixed axes	axes whose origin and orientation are fixed in a small material portion of the body which is such that the axes always remain mutually perpendicular. (The ones used are such that the origin coincides with the centre of gravity of the body, and the axes are parallel to the principal axes of inertia, during the datum motion.)
mean-body axes	axes, with origin at the centre of gravity, of the body, orientated so that the kinetic energy relative to the axes is a minimum.
no-deformation-body-fixed axes	an axes system, arbitrary except in-so-far as it is of the same order of nearness to the body-fixed axes and the datum-motion-body-fixed axes, with respect to which it is assumed, in the constant-velocity axes derivation, that the position of any particle of the body is given as a linear function of deformational generalised coordinates.
datum-attitude earth axes	axes with which the body-fixed axes coincide during the datum motion. (These are not in general body-fixed axes. They are earth-directed not earth-fixed.)

(iv) Orientation

angle of inclination	angle between the x-axis of the body-fixed axes and the horizontal plane
angle of bank	angle between the z-axis, of the body-fixed axes, and the vertical plane containing the x-axis of the same frame

* With one of the assumed forms of deformation (equation (1)) the body-fixed axes only remain mutually at right angles for small perturbations of the datum motion.

GLOSSARY OF TERMS (concluded)

(iv) (concluded)

nose-azimuth angle	angle between the projection of the x-axis, of the body-fixed axes, on the horizontal plane, and the x-axis of the normal earth-fixed axes

(v) Deformation

free-free modes	unconstrained modes of deformation - actually a slight constraint has been imposed in that the body-fixed axes are kept orthogonal during small deformations
encastré modes	modes of deformation constrained to have zero values and zero slopes at the reference point

LIST OF SYMBOLS

In this list i, j are used as typical dummy subscripts.

A_ϕ etc	skew-symmetric matrices involving ϕ, θ, ψ, etc (see equation (6))
A_{ij}	ponderous inertia coefficient
$B_{\phi\theta}$	lower triangular matrix involving ϕ, θ, ψ (see equation (56))
C_{ex}	lower triangular matrix involving e, f, g, x, y, z (see equation (71))
D	differential operator d/dt
E_0	strain energy when no perturbation
$-E_i$	generalised structural force
E_{ij}	structural stiffness coefficient
$-G_i$	generalised gravitational force
G_i	gyrostatic force
G_{ij}	gravitational stiffness coefficient
G^*_{ij}	gravitational 'build-up' coefficient (*cf* for example Table 5)
I	unit matrix
I_x, I_y, I_z	principal moments of inertia of undeformed aircraft
I_n	$= \mathrm{diag}\{I_x \; I_y \; I_z\}$
J_i	a certain coupling force between the rotational body freedoms and the deformational freedoms in Lagrange's equations referred to a non-inertial frame[15]
J_{ij}	ponderous damping coefficient
K_ϕ etc	matrices formed from the elements of $\{\phi \; \theta \; \psi\}$ etc (see equation (57a))
L	aerodynamic rolling moment
$\bar{L}$	total rolling moment
L_g, L_p, L_u	gravitational, propulsive, upholding (support) rolling moments
$L_x, L_{\dot{x}}, L_\phi, L_{\dot{\phi}}, L_u, L_p$ etc	aerodynamic rolling moment coefficients
M	aerodynamic pitching moment
$\bar{M}$	total pitching moment

LIST OF SYMBOLS (continued)

M_g, M_p, M_u	gravitational, propulsive, upholding (support) pitching moments
$M_x, M_{\dot{x}}, M_\phi, M_{\dot{\phi}}, M_u, M_p$ etc	aerodynamic pitching moment coefficients
N	aerodynamic yawing moment
$\bar{N}$	total yawing moment
N_g, N_p, N_u	gravitational, propulsive, upholding (support) yawing moments
$N_x, N_{\dot{x}}, N_\phi, N_{\dot{\phi}}, N_u, N_p$ etc	aerodynamic yawing moment coefficients
O	position of reference point (particle), origin of body-fixed axes
O_n	position of reference point when perturbations involve no deformation, origin of no-deformation-body-fixed axes (*cf* section 3)
Oxyz	body-fixed axes
$O_c x_c y_c z_c$	constant-velocity axes (identical with datum-attitude earth axes)
$O_m x_m y_m z_m$	mean-body axes
$O_n x_n y_n z_n$	no-deformation-body-fixed axes
$O_0 x_0 y_0 z_0$	normal earth-fixed axes
$-P_i$	generalised propulsive force
P_q	matrix of modal slopes at reference point (equation (11))
P_{ij}	propulsive stiffness coefficient
Q_i	generalised aerodynamic force
$\bar{Q}_i$	total generalised force
Q_ϕ	matrix relating angular velocities and orientation (*cf* equation (36)), see Ref 1 (Part II) or Ref 2)
Q_{ij}	aerodynamic coefficient
R	modal matrix (equation (2))
R_0	value of modal matrix at reference point
R_1, R_2, R_3	certain matrices, see equations (59) to (61), formed from the modal matrix elements
S	axes transformation matrix (*cf* equation (5), see Ref 1 (Part II) or Ref 2)
S_{Φ_f}	S where necessary to designate arguments - in this case Φ_f, Θ_f, Ψ_f

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LIST OF SYMBOLS (continued)

$-S_i$	generalised upholding (support) force
S_{ij}	upholding (support) stiffness coefficient
T_1, T_2, T_3	certain matrices whose elements are particular datum motion particle coordinates (see equations (62) to (64))
V_0	centrifugal potential function used in Lagrange's equations referred a non-inertial frame[15]
V_{ij}	ponderous stiffness coefficient
W	kinetic energy relative to reference frame
X	overall aerodynamic force resolute
$\bar{X}$	total overall force resolute
X_g, X_p, X_u	gravitational, propulsive, upholding (support) overall force resolutes
$X_x, X_{\dot{x}}, X_\phi, X_{\dot{\phi}}, X_u, X_p$ etc	aerodynamic force resolute coefficients
Y	overall aerodynamic force resolute
$\bar{Y}$	total overall force resolute
Y_g, Y_p, Y_u	gravitational, propulsive, upholding (support) overall force resolutes
$Y_x, Y_{\dot{x}}, Y_\phi, Y_{\dot{\phi}}, Y_u, Y_p$ etc	aerodynamic force resolute coefficients
Z	overall aerodynamic force resolute
$\bar{Z}$	total overall force resolute
Z_g, Z_p, Z_u	gravitational, propulsive, upholding (support) overall force resolutes
$Z_x, Z_{\dot{x}}, Z_\phi, Z_{\dot{\phi}}, Z_u, Z_p$ etc	aerodynamic force resolute coefficients
a_i	x-component of modal function (equation (2))
b_i	y-component of modal function (equation (2))
c_i	z-component of modal function (equation (2))
e	x-component of local aerodynamic force vector
e_i	coefficient in x-component of local aerodynamic force vector when near the ground

LIST OF SYMBOLS (continued)

e_i'	coefficient in x-component of local aerodynamic force vector when well away from the ground
e_g, e_p etc	x-components of gravitational, propulsive, etc local force vectors
e_{gi}, e_{si} etc	coefficients in x-components of gravitational, structural etc local force vectors
f	y-component of local aerodynamic force vector
f_i	coefficient in y-component of local aerodynamic foree vector when near the ground
f_i'	coefficient in y-component of local aerodynamic force vector when well away from the ground
f_g, f_p etc	y-components of gravitational, propulsive etc local force vectors
f_{gi}, f_{si} etc	coefficients in y-components of gravitational, structural etc local force vectors
g	acceleration due to gravity
g	z-component of local aerodynamic force vector
g_i	coefficient in z-component of local aerodynamic force reactor when near the ground
g_i'	coefficient in z-component of local aerodynamic force vector when well away from the ground
g_g, g_p etc	z-components of gravitational, propulsive etc local force vectors
g_{gi}, g_{si} etc	coefficients in z-components of gravitational, structural etc local force vectors
j_{Φ_f}	first column of S_{Φ_f} (equation (B-9))
k_{Φ_f}	second column of S_{Φ_f} (equation (B-10))
ℓ	direction cosine of outward drawn normal to surface
ℓ_{Φ_f}	third column of S_{Φ_f} (equation (39))
m	direction cosine of outward drawn normal to surface
m	mass of aircraft
n	direction cosine of outward drawn normal to surface
n	number of deformational degrees of freedom
p	angular velocity resolute
q	angular velocity resolute
q_i	generalised coordinate
r	angular velocity resolute
t	time
u	linear velocity resolute
u_m	particle velocity resolute

LIST OF SYMBOLS (continued)

v	linear velocity resolute
v_m	particle velocity resolute
w	linear velocity resolute
w_m	particle velocity resolute
x	particle position resolute
x_1	resolute of reference point position relative to its position in datum motion
x_c	particle position resolute relative to origin of constant-velocity axes
x_m	particle position resolute relative to origin of mean-body axes
x_n	particle position resolute relative to origin of no-deformation-body-fixed axes
y	particle position resolute
y_1	resolute of reference point position relative to its position in datum motion
y_c	particle position resolute relative to origin of constant-velocity axes
y_m	particle position resolute relative to origin of mean-body axes
y_n	particle position resolute relative to origin of no-deformation-body-fixed axes
z	particle position resolute
z_1	resolute of reference point position relative to its position in datum motion
z_c	particle position resolute relative to origin of constant-velocity axes
z_m	particle position resolute relative to origin of mean-body axes
z_n	particle position resolute relative to origin of no-deformation-body-fixed axes
$z_p(x_f,y_f,z_f) = z_p^{(0)}(x_0,y_0)$	
$z_p^{(0)}(x_0,y_0)$	value of z_0 at the surface of the ground when under no external load
Θ	angle of inclination
Φ	angle of bank
Ψ	nose-azimuth angle (also known as the heading or heading angle)

LIST OF SYMBOLS (continued)

$\delta_u(x_f,y_f,z_f)$	$= \delta_u^{(0)}(x_0,y_0)$
$\delta_u^{(0)}(x_0,y_0)$	earth (or support) stiffness coefficient
δm	mass of a particle
θ	orientation angle of body-fixed axes relative to datum motion
σ_{xy} etc	stress components
ϕ	orientation angle of body-fixed axes relative to datum motion
ψ	orientation angle of body-fixed axes relative to datum motion
ω_i	circular natural frequency
&	see Table 15

Dressings

(i) Subscripts

Absence of subscript, in appropriate cases, denotes values relative to or about the origin of the body-fixed axes.

c	denotes value relative to or about the origin of constant-velocity axes
f	denotes values during datum motion
g	denotes gravitational
m	denotes value relative to or about the origin of mean-body axes
n	denotes value relative to or about the origin of no-deformation-body-fixed axes
p	denotes propulsive, or (in Appendices) ground profile
s	denotes structural
u	denotes upholding (support)
0 (nought)	denotes value relative to or about the origin of normal earth-fixed axes

(ii) Superscripts

Absence of a bracketed superscript to a resolute indicates that it is the value of the resolute along the body-fixed axes.

(c)	denotes value of resolute along the datum-attitude earth axes (ie the constant-velocity axes). This is a change from the (f) of Ref 1
(m)	denotes value of resolute along the mean-body axes

LIST OF SYMBOLS (concluded)

(n)	denotes value of resolute along the no-deformation-body-fixed axes
T	denotes the transpose of a matrix
(0) (nought)	denotes value of resolute along normal earth-fixed axes

(iii) Suprascripts

• (dot)	denotes derivative with respect to time
- (bar)	denotes total or typical
^ (circumflex)	refers to body-fixed axes, encastré modes, displacement body freedoms
⌢ (cap)	refers to body-fixed axes, encastré modes, velocity body freedoms
~ (tilde)	refers to body-fixed axes, free-free modes, displacement body freedoms
⌣ (dip)	refers to body-fixed axes, free-free modes, velocity body freedoms
= (double bar)	refers to mean-body axes, free-free modes, velocity body freedoms

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17. Abstract

The equations of motion of an aircraft, for small perturbations from flight with constant linear and zero angular velocities, are developed in detail using:

constant-velocity	or	body-fixed axes;
encastré	or	free-free modes; and
displacement	or	velocity body freedom coordinates.

The relationship is clearly stated between these various forms; and with other proposed forms, in particular those using mean-body axes. The whole development is kept consistent, as far as possible, with the Hopkin notation scheme (R & M 3562).

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